SCATTERING THEORY IN QUANTUM MECHANICS

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C O N T E N T S

C O N T E N T S (Contd.)

INTRODUCTION

The aim of the present thesis is to present a review of certain topics in scattering theory which have been developed since 1950. Much of the work previous to this date is contained in text-books such as the well-known one by Mott and Massey.[1] In the first chapter we give an account of the formal theory of scattering as contained in the basic papers of Lippmann and Schwinger[2] and Gell-Mann and Goldberger,[3] and discuss some of the difficulties arising from the theory as developed therein, such as the validity of the various limiting processes encountered in the definition of the S-matrix (which was first introduced by Wheeler[4] in connection with nuclear reactions and was also discussed by Heisenberg[5] and Møller[6] in work performed prior to 1950). It will be seen that this general approach is largely mathematically unsatisfactory, due in a certain measure to unreal physical assumptions, and the second chapter is devoted to the reformulation of scattering theory in a rigorous mathematical manner, the original approach to which is due to two papers by Jauch[7, 8] which appeared in 1958. It will be seen how one is led to results exactly analogous to those obtained in the first chapter but which have rather more of an air of mathematical authenticity about them. Finally, in the third chapter, which is of quite a different character from the preceding ones we shall discuss, with reference to the non-relativistic scattering of a single particle by a central potential, two topics which are of particular interest in present-day field theory - dispersion relations and the Mandelstam representation.

CHAPTER I

FORMAL SCATTERING THEORY

1.1 Characterisation of a scattering system.

In the formal theory of scattering we assume that the energy operator H of the system under consideration can be split up into two parts $H = H_0 + H_I$ with the following properties:-

(i) H_0 has no discrete eigenvalues

(ii) the continuous eigenvalues of H are the same as the eigenvalues of H_0

(iii) the discrete eigenvalues of H are all smaller than the continuum values (1)

These three conditions are indeed satisfied for many systems in ordinary quantum mechanics, the separation being indeed even trivial in many cases. For example, if we have a single particle moving under the influence of an external potential, H_0 would merely be the kinetic energy operator for the particle, while H_I consisted of the potential energy.

In the case of quantum field theory, however, the situation is rather more complicated due to the fact that neither of conditions (ii) or (iii) are necessarily satisfied. Moreover the separation of H is rendered still more awkward since, because of self-interactions, the concept of a non-interacting system is rather obscure. Thus, by considering systems for which the above three conditions hold, it would appear that our results will not be applicable in the main to field theory without modifications.

This fact would seem to be ignored in much work, without leading to disastrous consequences.

The operator H_0 is considered to be such that if it were the entire Hamiltonian, the colliding parts of the system would have the same internal structure, but would experience no scattering due to the lack of interaction between them. H_I provides this interaction, hence causing scattering, and so our problem is to find the effect of H_I on the system.

1.2 Derivation of transition probabilities using the adiabatic hypothesis.

The Schrodinger equation for a scattering system is

$$(H_0 + H_I)|\Psi(t)\rangle = i\frac{\partial}{\partial t}|\Psi(t)\rangle \tag{2}$$

(with units $\hbar = c = 1$)

By transforming to the interaction picture we will ensure that the time dependence of the state vector associated with H_0 is removed. In this picture we have state vectors defined by

$$|\Psi'(t)\rangle = e^{iH_0 t}|\Psi(t)\rangle \tag{3}$$

and they satisfy the equation

$$H_I(t)|\Psi'(t)\rangle = i\frac{\partial}{\partial t}|\Psi'(t)\rangle \tag{4}$$

where $$H_I(t) = e^{iH_0t} H_I e^{-iH_0t} \tag{5}$$

With the definitions as above it is clearly seen that the interaction picture is the same as the Schrödinger picture at the time $t = 0$.

We now introduce the operator $U(t,t_0)$ which determines the time development of the state in the interaction picture:

$$|\Psi'(t)\rangle = U(t,t_0)|\Psi'(t_0)\rangle \tag{6}$$

This operator has three obvious properties:

(i) it is unitary

(ii) $U(t,t) = 1$

(iii) $U(t,t_0) = U(t,t')\,U(t',t_0)$ (7)

From the solution of the Schrödinger equation

$$|\Psi(t)\rangle = e^{-iH(t-t_0)}\,|\Psi(t_0)\rangle \tag{8}$$

and the definition of the interaction picture in equation (3) we obtain an explicit form of $U(t,t_0)$:-

$$U(t,t_0) = e^{iH_0t}\,e^{-iH(t-t_0)}\,e^{-iH_0t_0} \tag{9}$$

It is also easy to obtain a differential equation for

$\mathcal{U}(t,t_0)$ on differentiating equation (6) and using equation (4):

$$i\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = H_I(t)\,\mathcal{U}(t,t_0) \tag{10}$$

This differential equation immediately yields an integral equation which incorporates the "initial condition" (7)(ii)

$$\mathcal{U}(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t')\,\mathcal{U}(t',t_0)\,dt' \tag{11}$$

If we had differentiated equation (6) with respect to t_0 we should have obtained the integral equation

$$\mathcal{U}(t,t_0) = 1 + i\int_{t}^{t_0} \mathcal{U}(t,t')\,H_I(t')\,dt' \tag{12}$$

What we want to know is the relationship between the state in the distant past (i.e. $t_0 \to -\infty$) before scattering takes place, and the state in the distant future (i.e. $t \to +\infty$) after scattering has taken place. We can then find the probability for the system to be in any particular state after the scattering process. This, of course, points to our motivation for the introduction of the interaction picture; both in the distant past and the distant future we should expect the scattering parts of the system to be well separated, and thus H_0 to be the only effective part of the Hamiltonian and this means that the state vector in the interaction picture will be time independent both in the far past and in the distant future.

Now let us define operators $U(t,-\infty)$ and $U(\infty,t)$ by the relations

$$|\Psi'(t)\rangle = U(t,-\infty) \lim_{t_0 \to -\infty} |\Psi'(t_0)\rangle \tag{13}$$

and
$$\lim_{t_0 \to +\infty} |\Psi'(t_0)\rangle = U(\infty,t)\,|\Psi'(t)\rangle \tag{14}$$

It is clear that $U(t,-\infty)$ has a meaning only if its domain is restricted to those state vectors (in the interaction picture) which have a limit at $-\infty$; and $U(\infty,t)$ only if its range is restricted to the vectors with a limit at $+\infty$. Ekstein[9] showed that this condition was equivalent to the absence of any bound states in the wave packet.

It is important to note that entirely different operators are defined by

$$W(\infty,t) = \lim_{t' \to \infty} U(t',t)$$

$$W(t,-\infty) = \lim_{t' \to -\infty} U(t,t') \tag{15}$$

Of course, merely by looking at the explicit representation for the $\mathcal{U}$ operator given in equation (9), we see that we immediately come across the difficulty of giving a precise meaning to the above limiting process. What is done in effect is to require that W should be identical with $\mathcal{U}$ in the restricted range mentioned above (i.e. those state vectors having appropriate limits at $+\infty$ and $-\infty$).

However, with the definitions of $\mathcal{U}(t,-\infty)$ and $\mathcal{U}(\infty,t)$ as above, we are ensured that the following results will hold, analogous to equations (7) for finite arguments:-

(i) $\mathcal{U}(\infty,t)$ and $\mathcal{U}(t,-\infty)$ are unitary.

(ii) $\mathcal{U}(\infty,t) = \mathcal{U}(\infty,t')\,\mathcal{U}(t',t)$

(iii) $\mathcal{U}(t,-\infty) = \mathcal{U}(t,t')\,\mathcal{U}(t',-\infty)$ (16)

In addition we are ensured of the existence of $S \equiv \mathcal{U}(\infty,-\infty)$ by the defining relation

$$\lim_{t\to\infty} |\Psi'(t)\rangle = \mathcal{U}(\infty,-\infty) \lim_{t'\to-\infty} |\Psi'(t')\rangle \tag{17}$$

and this operator, too, will be unitary.

It follows immediately that the integral equations satisfied by these $\mathcal{U}$ operators with infinite arguments are as follows

$$\mathcal{U}(t,-\infty) = 1 - i\int_{-\infty}^{t} H_I(t')\,\mathcal{U}(t',-\infty)\,dt' \tag{18}$$

$$\mathcal{U}(\infty,t) = 1 - i\int_t^\infty H_I(t')\,\mathcal{U}(\infty,t')\,dt' \tag{19}$$

$$S \equiv \mathcal{U}(\infty,-\infty) = 1 - i\int_{-\infty}^{\infty} H_I(t')\,\mathcal{U}(t',-\infty)\,dt' \tag{20}$$

The S operator as defined above is the scattering operator: it generates the final state from the initial state. For definiteness let us introduce eigenfunctions $|\Phi_a\rangle$ for the separated parts of the system, so that eigenfunctions of this type will describe the initial and final states of the system. This means that if $|\Phi_a\rangle$ is the initial state, the final state will be $S|\Phi_a\rangle$ and so the probability of finding the system in the final state $|\Phi_b\rangle$ is given by

$$|\langle\Phi_b|S|\Phi_a\rangle|^2 = |S_{ba}|^2 \tag{21}$$

It is convenient to introduce the operator

$$T = S - 1 \tag{22}$$

which gives the change in the state vector due to the interaction. We can now say that the probability of finding the system in a final state $|\Phi_b\rangle$ differing from the initial state $|\Phi_a\rangle$ is given by

$$W_{ba} = |\langle \Phi_b | T | \Phi_a \rangle|^2 = |T_{ba}|^2 \qquad (23)$$

This quantity is known as the transition probability.

For future reference we shall note the important result that the unitarity of S implies the relationship

$$T^\dagger T = -(T + T^\dagger) \qquad (24)$$

(where the $\dagger$ refers to hermitian conjugate)

It is important to note that $|\Phi_a\rangle$ cannot be an exact eigenstate of H_0, since this would imply the momentum to be exactly known, and hence, by the uncertainty principle, we should have complete indeterminacy in position which is, of course, incompatible with the spatial localisation required by the definite separation of the parts of the system. What is required is a superposition of momentum states (i.e. a wave packet) and this method of approach will be outlined later in this chapter. An equivalent description is, according to Lippmann and Schwinger, obtained by the assumption that the $|\Phi_a\rangle$ are exact eigenstates of H_0 (i.e. plane waves) and postulating an adiabatic decrease in the interaction as $t \to \pm\infty$. This is effected by replacing H_I by $H_I e^{-\epsilon|t|}$ where ϵ may be arbitrarily small. We shall outline their theory and then indicate several objections to it.

The integral equations (18), (19), (20) become, on utilising the adiabatic switch-off procedure:

$$\mathcal{U}(t,-\infty) = 1 - i\int_{-\infty}^{t} H_I(t')\, e^{-\epsilon|t'|}\, \mathcal{U}(t',-\infty)\, dt' \tag{25}$$

$$\mathcal{U}(\infty,t) = 1 - i\int_{t}^{\infty} H_I(t')\, e^{-\epsilon|t'|}\, \mathcal{U}(\infty,t')\, dt' \tag{26}$$

$$S = 1 - i\int_{-\infty}^{\infty} H_I(t')\, e^{-\epsilon|t'|}\, \mathcal{U}(t',-\infty)\, dt' \tag{27}$$

Hence we obtain an expression for the elements of the T- matrix:

$$T_{ba} = -i\int_{-\infty}^{\infty} dt\, \langle \bar{\Phi}_b | H_I(t)\, \mathcal{U}(t,-\infty)\, e^{-\epsilon|t|} | \bar{\Phi}_a \rangle$$

$$= -i\int_{-\infty}^{\infty} dt\, \langle \bar{\Phi}_b | e^{iH_0 t} H_I e^{-iH_0 t}\, \mathcal{U}(t,-\infty)\, e^{-\epsilon|t|} | \bar{\Phi}_a \rangle$$

$$= -i\int_{-\infty}^{\infty} dt\, \langle \bar{\Phi}_b | H_I\, e^{i(E_b - H_0)t}\, \mathcal{U}(t,-\infty)\, e^{-\epsilon|t|} | \bar{\Phi}_a \rangle$$

using $\quad H_0 |\bar{\Phi}_b\rangle = E_b |\bar{\Phi}_b\rangle$

$$= -i\, \langle \bar{\Phi}_b | H_I | \bar{\Psi}_a^{(+)}(E_b) \rangle \tag{28}$$

where $$|\Psi_a^{(+)}(E)\rangle = \int_{-\infty}^{\infty} dt\, e^{i(E-H_0)t}\, e^{-\epsilon|t|}\, U(t,-\infty)|\Phi_a\rangle \tag{29}$$

A similar procedure can be carried out starting with the operators $U(-\infty,t)$ and $U(t,\infty)$ defined in analogous ways to $U(t,-\infty)$ and $U(\infty,t)$ and which satisfy, for instance, the integral equation

$$U(t,\infty) = 1 + i\int_{t}^{\infty} H_I(t')\, U(t',\infty)\, dt' \tag{30}$$

We can then define the operator $U(-\infty,\infty)$ which is obviously equivalent to S^{-1} and, using $S^{-1} - 1 = T^{\dagger}$ which follows from equation (22) and the unitarity of S we obtain eventually

$$T_{ab} = -i\langle \Psi_a^{(-)}(E_b)|H_I|\Phi_b\rangle \tag{31}$$

where

$$|\Psi_a^{(-)}(E)\rangle = \int_{-\infty}^{\infty} dt\, e^{i(E-H_0)t}\, e^{-\epsilon|t|}\, U(t,\infty)|\Phi_a\rangle \tag{32}$$

The integral equations for $U(t,-\infty)$ and $U(\infty,t)$ (equations (25) and (30)) will obviously yield integral equations for $|\Psi_a^{(+)}(E)\rangle$ and $|\Psi_a^{(-)}(E)\rangle$. These are, according to Lippmann and Schwinger,

$$|\Psi_a^{(+)}(E)\rangle = \int_{-\infty}^{\infty} dt\, e^{i(E-E_a)t} e^{-\epsilon|t|} |\Phi_a\rangle - i\int_0^{\infty} d\tau\, e^{i(E-H_0)\tau} e^{-\epsilon\tau} H_I |\Psi_a^{(+)}(E)\rangle \tag{33}$$

$$|\Psi_a^{(-)}(E)\rangle = \int_{-\infty}^{\infty} dt\, e^{i(E-E_a)t} e^{-\epsilon|t|} |\Phi_a\rangle + i\int_0^{\infty} d\tau\, e^{-i(E-H_0)\tau} e^{-\epsilon\tau} H_I |\Psi_a^{(-)}(E)\rangle \tag{34}$$

where $\tau = |t-t'|$

Performing the integration gives

$$|\Psi_a^{(\pm)}(E)\rangle = 2\pi\delta(E-E_a)|\Phi_a\rangle + \frac{1}{E-H_0 \pm i\epsilon} H_I |\Psi_a^{(\pm)}(E)\rangle \tag{35}$$

Writing $$|\Psi_a^{(\pm)}(E)\rangle = 2\pi\,\delta(E-E_a)\,|\Psi_a^{(\pm)}\rangle \tag{36}$$

we obtain the relationship

$$|\Psi_a^{(\pm)}\rangle = |\Phi_a\rangle + \frac{1}{E_a-H_0 \pm i\epsilon} H_I |\Psi_a^{(\pm)}\rangle \tag{37}$$

These are the well-known Lippmann-Schwinger equations for the outgoing and incoming scattered waves associated with the plane waves $|\Phi_a\rangle$

If we now write

$$T_{ba} = -2\pi i\,\delta(E_a - E_b)\,\mathcal{J}_{ba} \tag{38}$$

we obtain

$$\mathcal{J}_{ba} = \langle \Phi_b | H_I | \Psi_a^{(+)} \rangle = \langle \Psi_b^{(-)} | H_I | \Phi_a \rangle \tag{39}$$

$\mathcal{J}$ is known as the association matrix, and its elements are defined only when taken between states of equal energy.

The transition probability is now given by (using equation (23))

$$W_{ba} = 4\pi^2 \left[\delta(E_a - E_b) \right]^2 |\mathcal{J}_{ba}|^2$$

$$= 4\pi^2 \delta(E_a - E_b)\,\delta(0)\,|\mathcal{J}_{ba}|^2 \tag{40}$$

The factor $\delta(0)$ comprises one of the difficulties of this approach to scattering theory; it can however be interpreted in an intuitive manner as follows:-

We write the δ-function in its integral form

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt}\, dt \tag{41}$$

$$\delta(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i0.t}\, dt$$

and so $$W_{ba} = 2\pi\,\delta(E_a - E_b)\,|J_{ba}|^2 \int_{-\infty}^{\infty} dt \tag{42}$$

and this is interpreted by saying that the transition probability per unit time (i.e. the transition rate) is given by

$$\omega_{ba} = 2\pi\,\delta(E_a - E_b)\,|J_{ba}|^2 \tag{43}$$

(We should note that this result can also be derived in a slightly more satisfactory manner from the relationship

$$\omega_{ba} = \frac{\partial}{\partial t}\,|\langle \Phi_b | U(t,-\infty) | \Phi_a \rangle|^2 \tag{44}$$

which merely states that the transition rate is the increase, per unit time, of the probability of finding the system in the state $|\Phi_b\rangle$, when it was known initially to be in the state $|\Phi_a\rangle$).

1.3 The connection between cross-section and transition probability; the optical theorem.

From the transition rate we may derive an expression for the quantity which is usually considered experimentally - the cross-section (either total or differential).

Now the cross section is defined by the relationship

$$\text{cross-section} = \frac{\text{transition rate x density of final states}}{\text{flux of initial states}} \tag{45}$$

We are, of course, considering here the simple case of single channel scattering and can consider the initial state characterised by a plane wave state with momentum $\underline{k}_a$ and the final state by the plane wave state with momentum $\underline{k}_b$.

Then we are interested in the transition rate into the range $d^3\underline{k}_b$ about the vector $\underline{k}_b$. This rate is, from equation (43) given by

$$\text{Transition rate} = 2\pi\, \delta(E_b - E_a)\, |J_{ba}|^2\, d^3\underline{k}_b \tag{46}$$

The density of final states is (assuming a normalisation of one particle per volume $(2\pi)^3$) given by[*]

$$\rho(\underline{k}_b) = 1 \tag{47}$$

The flux of incident states is, assuming the same normalisation:-

* See reference (10) page 194.

$$\text{Flux} = \frac{V_a}{(2\pi)^3} \tag{48}$$

where V_a is the velocity of the incident particles.

It then follows, from combining equations (45), (46), (47) and (48), and writing

$$d^3\underline{k}_b = k_b^2\, dk_b\, d\Omega \tag{49}$$

where $d\Omega$ is an element of solid angle, and noting that the differential cross-section $d\sigma$ is given by integrating over all possible values of k_b that we obtain

$$d\sigma = \frac{(2\pi)^4}{V_a}\int |J_{ba}|^2 \delta(E_b - E_a)\, k_b^2 \frac{dk_b}{dE_b}\, dE_b\, d\Omega$$

$$= (2\pi)^4 |J_{ba}|^2 \frac{k_a^2}{V_a^2}\, d\Omega \tag{50}$$

From the unitarity condition in equation (24) we obtain, on taking a matrix element and substituting equation (38), the well-known optical theorem:

$$4\pi^2 \sum_b \delta(E_a - E_b) J_{ba}^* \delta(E_b - E_c) J_{bc} = 2\pi i\, \delta(E_a - E_c)(J_{ac} - J_{ca}^*) \tag{51}$$

Cancelling $\delta(E_a - E_c)$ and then taking the special case

$a = c$ gives

$$2\pi \sum_b \delta(E_a - E_b) |J_{ba}|^2 = -2\, \mathrm{Im}\, J_{aa}$$

i.e. $$\sum_b w_{ba} = -2\, \mathrm{Im}\, J_{aa} \tag{52}$$

1.4 Criticism of the adiabatic hypothesis.

We must now look at the arguments by which Lippmann and Schwinger arrive at the above results. There seem to be two main points to be made.

(i) As pointed out by Ekstein[9] the existence of the operators $U(t, -\infty)$ and $U(\infty, t)$ is guaranteed only for those systems which do not possess bound states.

(ii) As indicated by Sunakawa[11] equations (33) and (34) by no means follow obviously from the integral equations (18) and (30). Indeed substituting the integral equation

$$U(t, -\infty) = 1 - i \int_{-\infty}^{t} H_I(t')\, U(t', -\infty)\, dt' \tag{18}$$

into equation (29) gives

$$|\Psi_a^{(+)}(\epsilon)\rangle = \int_{-\infty}^{\infty} dt\, e^{i(\epsilon-\epsilon_a)t}\, e^{-\epsilon|t|}\, |\Phi_a\rangle - i\int_0^{\infty} d\tau\, e^{i(\epsilon-H_0)\tau} \times$$

$$\times H_I \int_{-\infty}^{\infty} dt'\, e^{-\epsilon|t'+\tau|}\, e^{i(\epsilon-H_0)t'}\, U(t',-\infty)\,|\Phi_a\rangle \quad (53)$$

instead of equation (33).

Of course the adiabatic "switch-off" led to the integral equation (25) instead of (18), viz.

$$U(t,-\infty) = 1 - i\int_{-\infty}^{t} H_I(t')\,U(t',-\infty)\,dt' \quad (25)$$

and this equation was made the basis of attempts by several people to derive the Lippmann-Schwinger equations by an iteration method. However the series obtained by iteration will not necessarily converge when there is the possibility of the existence of bound states. In addition one arrives at the conclusion that the eigenfunctions of the total Hamiltonian belonging to the continuum constitute a complete set on their own without the necessity of any contribution from the bound states.

From all these considerations it is readily seen that the validity of the adiabatic "switch-off" procedure becomes rather doubtful, especially when applied to systems which admit bound states.

1.5 Derivation of transition probability using an averaging over states.

An alternative method of approach was suggested by Gell-Mann and Goldberger.[3] In their treatment they suggest that we should consider the manner in which the state $|\Psi(t)\rangle$ (in the Schrodinger picture) has been prepared. We might try, for instance, the model in which at some time t_0 in the distant past the system was in the free state $|\Phi_a\rangle$

$$\text{i.e.} \qquad |\Psi_a(t)\rangle = e^{-iH(t-t_0)}|\Phi_a\rangle \tag{54}$$

where, of course, the time dependence of the free states is given by

$$|\Phi_a(t_0)\rangle = e^{-iE_a t_0}|\Phi_a\rangle \tag{55}$$

$$\text{with} \qquad H_0|\Phi_a\rangle = E_a|\Phi_a\rangle \tag{56}$$

and the $|\Phi_a\rangle$ are normalised in a large box in the usual manner.

However it is found that it is more convenient to replace this rather unphysical assumption that the train of incident waves is released all at once at time t_0, by the representation that it is fed in over a period of time in the past, i.e. $|\Psi_a(t)\rangle$ will be an average over t_0 of equation (54). For example, we might take

$$|\Psi_a(t)\rangle = \frac{1}{\tau}\int_{-\tau}^{0} dt_0 \, e^{-iH(t-t_0)} |\Phi_a(t_0)\rangle \tag{57}$$

where τ can later be allowed to tend to $+\infty$.

It is convenient to use an alternative form, which is completely equivalent to equation (57) but is slightly more advantageous mathematically:-

$$|\Psi_a^{(\epsilon)}(t)\rangle = \epsilon\int_{-\infty}^{0} dt_0 \, e^{\epsilon t_0} e^{-iH(t-t_0)} |\Phi_a(t_0)\rangle \tag{58}$$

where ϵ will later be allowed to tend to 0.

Transforming to the interaction representation we obtain

$$|\Psi_a'^{(\epsilon)}(t)\rangle = e^{iH_0 t} e^{-iHt} \epsilon\int_{-\infty}^{0} dt_0 \, e^{\epsilon t_0} e^{iHt_0} e^{-iH_0 t_0} |\Phi_a\rangle$$

$$= \epsilon\int_{-\infty}^{0} dt_0 \, e^{\epsilon t_0} \, U(t,t_0) |\Phi_a\rangle \tag{59}$$

using equation (9).

We now note the following result:-

Define

$$\underset{t_0\to-\infty}{L} f(t_0) \equiv \lim_{\epsilon\to 0} \epsilon\int_{-\infty}^{0} dt_0 \, e^{\epsilon t_0} f(t_0) \tag{60}$$

Then if $f(t_0)$ possesses a limit as $t_0 \to -\infty$ the L operation will give this limit, whereas if $f(t_0)$ oscillates as $t_0 \to -\infty$ the L process will give 0.

So we see that a very convenient definition of $U(t,-\infty)$ is given by:-

$$U(t,-\infty) = \lim_{\epsilon\to 0} \epsilon \int_{-\infty}^{0} dt_0 \, e^{\epsilon t_0} U(t,t_0) \tag{61}$$

In an exactly analogous manner we define

$$U(\infty,t) = \lim_{\epsilon\to 0} \epsilon \int_{0}^{\infty} dt_0 \, e^{-\epsilon t_0} U(t_0,t) \tag{62}$$

These definitions ensure that one can carry over certain properties of the U operator with finite arguments to those with infinite arguments,

$$\text{e.g.} \quad U(\infty,t) = U(\infty,t')\,U(t',t)$$
$$U(t,-\infty) = U(t,t')\,U(t',-\infty) \tag{63}$$

In particular, the integral equation still holds for infinite arguments:

$$U(t,-\infty) = 1 - i\int_{-\infty}^{t} H_I(t')\,U(t',-\infty)\,dt' \tag{64}$$

Now the S -matrix is given by $U(\infty, -\infty)$ if a precise meaning has been given to the necessary limiting process involved. According to Gell-Mann and Goldberger it is immaterial whether we define S by applying the two limiting processes (61) and (62) to $U(t, t_0)$, or by applying to equation (64) a limiting process to give the required meaning to the oscillatory integrals. This procedure gives

$$S \equiv U(\infty, -\infty) = 1 - i\int_{-\infty}^{\infty} dt' \, H_I(t') \, U(t', -\infty) \tag{65}$$

Now

$$U(t, -\infty) = \lim_{\epsilon \to 0} \epsilon \int_{-\infty}^{0} dt_0 \, e^{\epsilon t_0} \, e^{iH_0 t} \, e^{-iH(t-t_0)} \, e^{-iH_0 t_0}$$

using equations (61) and (9)

$$= e^{iH_0 t} \, e^{-iHt} \lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon + i(H - H_0)} \tag{66}$$

Then, using the completeness relation

$$\sum_a |\Phi_a\rangle\langle\Phi_a| = 1 \tag{67}$$

we obtain

$$\mathcal{U}(t,-\infty) = e^{iH_0t} e^{-iHt} \lim_{\epsilon \to 0} \sum_a \frac{\epsilon}{\epsilon + i(H-E_a)} |\Phi_a\rangle\langle\Phi_a| \tag{68}$$

and hence

$$\mathcal{U}(0,-\infty) = \sum_a |\Psi_a^{(+)}\rangle\langle\Phi_a| \tag{69}$$

where
$$|\Psi_a^{(+)}\rangle = \lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon + i(H-E_a)} |\Phi_a\rangle \tag{70}$$

which can be rewritten to give

$$|\Psi_a^{(+)}\rangle = |\Phi_a\rangle + \lim_{\epsilon \to 0} \frac{1}{E_a - H_0 + i\epsilon} H_I |\Psi_a^{(+)}\rangle \tag{71}$$

i.e. the Lippmann-Schwinger equation.

Equation (69) gives

$$\Omega_+ |\Phi_a\rangle = |\Psi_a^{(+)}\rangle \tag{72}$$

where $\Omega_+ \equiv \mathcal{U}(0,-\infty)$ is known as Møller's wave-matrix, i.e. the operator which produces the outgoing scattered state from the plane wave state.

In an exactly analogous manner $\Omega_- \equiv \mathcal{U}(0,\infty)$ produces the incoming scattered state:

$$\Omega_- |\Phi_a\rangle = |\Psi_a^{(-)}\rangle$$

$$= \lim_{\epsilon \to 0} \frac{\epsilon}{\epsilon - i(H - E_a)} |\Phi_a\rangle \tag{73}$$

and $|\Psi_a^{(-)}\rangle$ satisfies the Lippmann-Schwinger equation

$$|\Psi_a^{(-)}\rangle = |\Phi_a\rangle + \lim_{\epsilon \to 0} \frac{1}{E_a - H_0 - i\epsilon} H_I |\Psi_a^{(-)}\rangle \tag{74}$$

Now, according to Gell-Mann and Goldberger, the above definitions of $U(\infty, -\infty)$, $U(\infty, 0)$ and $U(0, -\infty)$ imply the relationship

$$U(\infty, -\infty) = U(\infty, 0)\, U(0, -\infty) \tag{75}$$

The fact that $U(t, t_0)\, U(t_0, t) = 1$ and the unitarity of U gives immediately that

$$U(t, t_0) = U(t_0, t)^\dagger \quad \text{for finite times} \tag{76}$$

Applying either of the limiting processes (61) or (62) will yield the corresponding results for infinite arguments:

$$U(\pm\infty, 0) = U(0, \pm\infty)^\dagger \tag{77}$$

So
$$U(\infty, 0) = \Omega_-^\dagger \tag{78}$$

and
$$S = \Omega_-^\dagger \Omega_+ \tag{79}$$

The matrix elements of the S operator are then given by

$$S_{ba} = \langle \Phi_b | S | \Phi_a \rangle$$

$$= \langle \Phi_b | \Omega_-^+ \Omega_+ | \Phi_a \rangle$$

$$= \langle \Omega_- \Phi_b | \Omega_+ \Phi_a \rangle$$

$$= \langle \Psi_b^{(-)} | \Psi_a^{(+)} \rangle \tag{80}$$

An alternative form is obtained from equation (65). We have, using equation (69) and the fact (which may easily be verified) that

$$H | \Psi_a^{(+)} \rangle = E_a | \Psi_a^{(+)} \rangle \tag{81}$$

$$U(t,-\infty) = e^{iH_0 t} \sum_a e^{-iE_a t} | \Psi_a^{(+)} \rangle \langle \Phi_a | \tag{82}$$

Hence $H_I(t)\, U(t,-\infty) = e^{iH_0 t} H_I \sum_a e^{-iE_a t} | \Psi_a^{(+)} \rangle \langle \Phi_a |$

$$= \sum_{a,b} | \Phi_b \rangle \langle \Phi_b | e^{i(E_b - E_a)t} H_I | \Psi_a^{(+)} \rangle \langle \Phi_a | \tag{83}$$

on insertion of a complete set of states.

Substituting into equation (65) gives

$$S = 1 - \sum_{a,b} |\Phi_b\rangle\langle\Phi_b|H_I|\Psi_a^{(+)}\rangle\langle\Phi_a| 2\pi i\,\delta(E_a - E_b) \tag{84}$$

and hence

$$S_{ba} = \delta(b-a) - 2\pi i\,\delta(E_a - E_b)\langle\Phi_b|H_I|\Psi_a^{(+)}\rangle \tag{85}$$

and so with $\qquad S = 1 + T \qquad (22)$

and $\qquad T_{ba} = -2\pi i\,\delta(E_a - E_b) J_{ba} \qquad (38)$

we obtain $\qquad J_{ba} = \langle\Phi_b|H_I|\Psi_a^{(+)}\rangle \qquad (86)$

the same result as was obtained by Lippmann and Schwinger's approach (equation (39)).

The unitarity of the S-matrix may be shown quite readily using the methods of Gell-Mann and Goldberger. We need the results

$$\Omega_+^\dagger \Omega_+ = \Omega_-^\dagger \Omega_- = 1 \tag{87}$$

$$\Omega_+ \Omega_+^\dagger = \Omega_- \Omega_-^\dagger = 1 - \sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha| \tag{88}$$

where the $|\Psi_\alpha\rangle$ are the bound states of the total Hamiltonian.

These follow simply from the definitions (72) and (73):

$$\Omega_+ = \sum_a |\Psi_a^{(+)}\rangle\langle\Phi_a| \tag{72}$$

$$\Omega_- = \sum_a |\Psi_a^{(-)}\rangle\langle\Phi_a| \tag{73}$$

and orthonormality and completeness relations (remembering that the complete set of eigenstates of H is the set $|\Psi_\alpha\rangle$ together with either the set $|\Psi_a^{(+)}\rangle$ or the set $|\Psi_a^{(-)}\rangle$)

Then $S^\dagger S = \Omega_+^\dagger \Omega_- \Omega_-^\dagger \Omega_+$ from equation (79)

$$= \Omega_+^\dagger \left(1 - \sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha|\right)\Omega_+$$

$$= 1 - \Omega_+^\dagger \sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha| \Omega_+ \tag{89}$$

Now the second term in the above expression is effectively zero since the Ω_+ operating on a state $|\Phi_a\rangle$ (which is the type of state used to obtain matrix elements of S) produces the state $|\Psi_a^{(+)}\rangle$ which is orthogonal to all the bound states $|\Psi_\alpha\rangle$. Then equation (89) reduces to $S^\dagger S = 1$, and in a similar manner we may show that $S S^\dagger = 1$.

1.6 Criticism of the averaging procedure.

Thus we have seen that the theory of Gell-Mann and Goldberger provides us with the Lippmann-Schwinger equations in a more satisfactory manner than the previous method, and we also encounter no difficulty in connection with the possible existence of bound states; in this respect the theory based on the averaging procedure is preferable to that based on the adiabatic switch-off. However there is another point which causes some difficulty: in the paper by Gell-Mann and Goldberger it is stated that $U(\infty,-\infty)$ as defined by equation (65) is equivalent to $U(\infty,0)\,U(0,-\infty)$ where $U(\infty,0)$ and $U(0,-\infty)$ are defined by equations (61) and (62). However, as Sunakawa(11) pointed out, if we substitute equations (61) and (62) into $U(\infty,0)\,U(0,-\infty)$ it is found that we are left with an extra term as compared with the required expression (65) for S. This means that we cannot definitely identify the two forms of S :-

$$S = U(\infty,-\infty)$$

and
$$S = U(\infty,0)\,U(0,-\infty)$$

and so, for example, the consistency of the expressions

$$S_{ba} = \langle \Psi_b^{(-)} | \Psi_a^{(+)} \rangle$$

and
$$T_{ba} = \langle \Phi_b | H_I | \Psi_a^{(+)} \rangle$$

must be in some doubt.

We hence reach the conclusion that the averaging procedure, too, requires some modification.

1.7 The theory on the basis of a new limiting process.

Sunakawa proposes new definitions for the $\mathcal{U}$ operators for infinite arguments, based on the integral equations (11) and (12) for finite times

$$\mathcal{U}(\pm\infty, t) = 1 - i \lim_{\epsilon \to 0} \int_t^{\pm\infty} dt' \, e^{-\epsilon|t'|} H_I(t') \, \mathcal{U}(t', t) \tag{90}$$

$$\mathcal{U}(t, \mp\infty) = 1 + i \lim_{\epsilon \to 0} \int_t^{\mp\infty} dt' \, e^{-\epsilon|t'|} \, \mathcal{U}(t, t') H_I(t') \tag{91}$$

where $\mathcal{U}(t, t')$ is given by the explicit expression (9) and the limit is to be taken after all other calculations. Although these expressions contain an $e^{-\epsilon|t'|}$ term, they are, of course, different from equations (25) and (26) based on the adiabatic switch-off.

As in previous cases the following properties are easily verified:-

(i) $\mathcal{U}(\pm\infty,t) = \mathcal{U}(\pm\infty,t')\,\mathcal{U}(t',t)$

(ii) $\mathcal{U}(t,\pm\infty) = \mathcal{U}(t,t')\,\mathcal{U}(t',\pm\infty)$

(iii) $\mathcal{U}(\pm\infty,t)^{\dagger} = \mathcal{U}(t,\pm\infty)$

(iv) $\mathcal{U}(t,\pm\infty)^{\dagger} = \mathcal{U}(\pm\infty,t)$ (92)

Also

$$\mathcal{U}(0,-\infty) = 1 + i \lim_{\epsilon\to 0} \int_0^{-\infty} e^{-\epsilon|t'|}\, e^{iHt'}\, e^{-iH_0t'}\, H_I(t')\,dt'$$

$$= 1 - i \lim_{\epsilon\to 0} \int_{-\infty}^{0} e^{\epsilon t'}\, e^{iHt'}\, H_I\, e^{-iH_0t'}\,dt'$$

$$= 1 - i \lim_{\epsilon\to 0} \sum_a \int_{-\infty}^{0} e^{\epsilon t'}\, e^{i(H-E_a)t'}\,dt'\; H_I |\Phi_a\rangle\langle\Phi_a|$$

inserting a complete set of states

$$= 1 - i \lim_{\epsilon\to 0} \sum_a \frac{1}{\epsilon + i(H-E_a)}\, H_I |\Phi_a\rangle\langle\Phi_a|$$

$$= \sum_a \left\{ |\Phi_a\rangle + \lim_{\epsilon\to 0} \frac{1}{E_a - H + i\epsilon}\, H_I |\Phi_a\rangle \right\} \langle\Phi_a| \qquad (93)$$

i.e. $$\mathcal{U}(0,-\infty) = \sum_a |\Psi_a^{(+)}\rangle\langle\Phi_a| \qquad (94)$$

where $$|\Psi_a^{(+)}\rangle = |\Phi_a\rangle + \lim_{\epsilon\to 0} \frac{1}{E_a - H + i\epsilon}\, H_I |\Phi_a\rangle \qquad (95)$$

or, by rearrangement

$$|\Psi_a^{(+)}\rangle = |\Phi_a\rangle + \lim_{\epsilon\to 0} \frac{1}{E_a - H_0 + i\epsilon} H_I |\Psi_a^{(+)}\rangle \qquad (96)$$

which is, once again, the Lippmann-Schwinger equation.

It hence follows that we can identify $U(0,-\infty)$ as defined above with Ω_+ as defined by equation (72):-

$$\Omega_+ |\Phi_a\rangle = |\Psi_a^{(+)}\rangle \qquad (72)$$

In an exactly analogous manner it follows that we can identify $U(0,\infty)$ with Ω_- where

$$\Omega_- |\Phi_a\rangle = |\Psi_a^{(-)}\rangle$$

and $|\Psi_a^{(-)}\rangle$ satisfies the familiar Lippmann-Schwinger equation.

It is now possible to define $U(\infty,-\infty)$ in two ways, namely by letting $t\to-\infty$ in equation (90) or $t\to+\infty$ in equation (91):-

$$U(\infty,-\infty) = 1 - i \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} H_I(t')\, U(t',-\infty) \qquad (97)$$

$$U(\infty,-\infty) = 1 + i \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} U(\infty,t')\, H_I(t') \qquad (98)$$

The equivalence of these two expressions is not immediately obvious, but may be proved quite simply. Moreover it can also be shown that

$$U(\infty,-\infty) = U(\infty,t)\,U(t,-\infty) \tag{99}$$

thus ensuring the equivalence of the definitions above and the product $U(\infty,0)\,U(0,-\infty)$ (which was the property that could not be assured in Gell-Mann and Goldberger's treatment.)

So, as in the previous treatment, we have for the S - matrix elements

$$\begin{aligned} S_{ba} &= \langle \Phi_b | S | \Phi_a \rangle \\ &= \langle \Phi_b | U(\infty,-\infty) | \Phi_a \rangle \\ &= \langle \Phi_b | U(\infty,0)\, U(0,-\infty) | \Phi_a \rangle \\ &= \langle \Psi_b^{(-)} | \Psi_a^{(+)} \rangle \end{aligned} \tag{100}$$

These elements may also be calculated from equation (97) above:-

$$S_{ba} = \langle \Phi_b | U(\infty,-\infty) | \Phi_a \rangle$$

$$= \langle \Phi_b | \Phi_a \rangle - i \langle \Phi_b | \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} H_I(t')\, U(t',0)\, U(0,-\infty) | \Phi_a \rangle$$

$$= \delta(b-a) - i \langle \Phi_b | \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} e^{iE_b t'} H_I e^{-iHt'} | \Psi_a^{(+)} \rangle$$

$$= \delta(b-a) - i \langle \Phi_b | \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dt' e^{-\epsilon|t'|} e^{i(E_b - E_a)t'} H_I | \Psi_a^{(+)} \rangle$$

$$= \delta(b-a) - 2\pi i \delta(E_b - E_a) \langle \Phi_b | H_I | \Psi_a^{(+)} \rangle \tag{101}$$

which is the same form as before

1.8 The theory based on the wave packet.

So by the use of the limiting processes (90) and (91) we have succeeded in deriving the Lippmann-Schwinger equations, and have obtained two equivalent expressions for **S**-matrix elements. However, the treatment so far, based on mathematical limiting processes rather far removed from any physical interpretation is felt to be rather artificial. This difficulty arises, of course, because we assume a plane wave state both for the initial and final configurations of the system. In a more realistic treatment by Hack[12] it is assumed that the initial state is of the form of a wave packet $|\bar{\Phi}_i\rangle$ which consists of a superposition of eigenfunctions

$|\Phi_a\rangle$ of the free Hamiltonian over a small range of eigenvalues, and at the end of the calculation the limit $|\bar{\Phi}_i\rangle \rightarrow |\Phi_a\rangle$ is taken, and we obtain all the previous equations. Hack's procedure is as follows:-

We start off with the Lippmann-Schwinger equations

$$|\Psi_a^{(\pm)}\rangle = |\Phi_a\rangle + \lim_{\epsilon \to 0} \frac{1}{E_a - H_0 \pm i\epsilon} H_I |\Psi_a^{(\pm)}\rangle \tag{102}$$

which, of course, give two sets (outgoing and incoming waves) of eigenfunctions of H with continuous eigenvalues E_a

$$H|\Psi_a^{(\pm)}\rangle = E_a|\Psi_a^{(\pm)}\rangle \tag{103}$$

In addition we may have the discrete bound states $|\Psi_\alpha\rangle$

$$H|\Psi_\alpha\rangle = E_\alpha|\Psi_\alpha\rangle \tag{104}$$

and we shall assume the conditions (1) to hold. What we are interested in is the convergence as $t \rightarrow -\infty$ of

$$U(0,t)|\Phi_a\rangle = e^{iHt}e^{-iE_at}|\Phi_a\rangle \tag{105}$$

We now expand $|\Phi_a\rangle$ in terms of a complete set of eigenstates of the total Hamiltonian:

$$|\Phi_a\rangle = \int dc\,|\Psi_c^{(+)}\rangle\langle\Psi_c^{(+)}|\Phi_a\rangle + \sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha|\Phi_a\rangle \tag{106}$$

and so $$\mathcal{U}(0,t)\,|\Phi_a\rangle = |\gamma_a(t)\rangle + |\beta_a(t)\rangle \tag{107}$$

where
$$|\beta_a(t)\rangle = \sum_\alpha e^{i(E_\alpha - E_a)t}\,|\Psi_\alpha\rangle\langle\Psi_\alpha|\Phi_a\rangle$$

and
$$|\gamma_a(t)\rangle = \int dc\; e^{i(E_c-E_a)t}\,|\Psi_c^{(+)}\rangle \lim_{\epsilon\to 0}\left\{\langle\Phi_c| + \left\langle\frac{1}{E_c-H_0+i\epsilon}H_I\Psi_c^{(+)}\right|\right\}|\Phi_a\rangle$$

$$= \int dc\; e^{i(E_c-E_a)t}\,|\Psi_c^{(+)}\rangle\left\{\delta(a-c) + \lim_{\epsilon\to 0}\left\langle H_I\Psi_c^{(+)}\right|\frac{1}{E_c-E_a-i\epsilon}|\Phi_a\rangle\right\}$$

$$= |\Psi_a^{(+)}\rangle + |\rho_a^{(+)}(t)\rangle \tag{108}$$

and it may now be shown quite simply that

$$\lim_{t\to-\infty}|\rho_a^{(+)}(t)\rangle = 0 \tag{109}$$

However the contribution from the bound states $|\beta_a(t)\rangle$ will oscillate as $t\to-\infty$ unless all the $\langle\Psi_\alpha|\Phi_a\rangle$ vanish. So when the total Hamiltonian H admits the possibility of bound states the limit (105) will not exist in general. This is just the result that was pointed out by Ekstein.[9]

If, however, we take as the initial state a wave packet $|\bar{\Phi}_i\rangle$:

$$|\bar{\Phi}_i\rangle = \int da\,|\Phi_a\rangle\langle\Phi_a|\bar{\Phi}_i\rangle \tag{110}$$

we have

$$\mathcal{U}(0,t)\,|\bar{\Phi}_i\rangle = \int da\,\langle\bar{\Phi}_a|\bar{\Phi}_i\rangle\left\{|\Psi_a^{(+)}\rangle + |\rho_a^{(+)}(t)\rangle + |\beta_a(t)\rangle\right\}$$

$$\equiv |\overline{\Psi_i^{(+)}}\rangle + |\overline{\rho_i^{(+)}(t)}\rangle + |\overline{\beta_i(t)}\rangle \qquad (111)$$

The introduction of an integration over the a's ensures that the limit of $|\overline{\beta_i(t)}\rangle$ vanishes as $t \to -\infty$ because of the factor $e^{i(E_\alpha - E_a)t}$ (essentially the Riemann-Lebesgue lemma). Also, because of equation (109) we must have

$$\lim_{t\to-\infty} |\overline{\rho_i^{(+)}(t)}\rangle = 0 \qquad (112)$$

so that $\displaystyle\lim_{t\to-\infty} \mathcal{U}(0,t)\,|\bar{\Phi}_i\rangle = |\overline{\Psi_i^{(+)}}\rangle$ (113)

Now the transition rate, ω_{ba}, is given by

$$\omega_{ba} = \lim_{|\bar{\Phi}_i\rangle \to |\bar{\Phi}_a\rangle}\ \lim_{t_0\to-\infty} \frac{\partial}{\partial t}\left|\langle\bar{\Phi}_b|\mathcal{U}(t,t_0)|\bar{\Phi}_i\rangle\right|^2 \qquad (114)$$

So we want to calculate

$$\frac{\partial}{\partial t}\left|\langle\bar{\Phi}_b|\mathcal{U}(t,0)\,\mathcal{U}(0,t_0)|\bar{\Phi}_i\rangle\right|^2$$

which, by equations (9) and (11), is given by

$$2\,\mathrm{Re}\Big\{-i\Big(\langle\Phi_b|e^{iH_0t}H_I e^{-iHt}\big[|\overline{\Psi_i^{(+)}}\rangle+|\overline{\rho_i^{(+)}(t_0)}\rangle+|\overline{\beta_i(t_0)}\rangle\big]\Big)\times$$

$$\times\Big(\langle\Phi_b|e^{iH_0t}e^{-iHt}\big[|\overline{\Psi_i^{(+)}}\rangle+|\overline{\rho_i^{(+)}(t_0)}\rangle+|\overline{\beta_i(t_0)}\rangle\big]\Big)^*\Big\} \qquad (115)$$

By the definition in equation (111) of $|\overline{\Psi_i^{(+)}}\rangle$ we have

$$\langle\overline{\Phi}_b|e^{iH_0t}e^{-iHt}|\overline{\Psi_i^{(+)}}\rangle=\langle\Phi_b|e^{iE_bt}\int da\,\langle\Phi_a|\overline{\Phi}_i\rangle e^{-iE_at}|\Psi_a^{(+)}\rangle$$

$$=\int da\, e^{i(E_b-E_a)t}\langle\Phi_a|\overline{\Phi}_i\rangle\langle\Phi_b|\Psi_a^{(+)}\rangle \qquad (116)$$

and hence

$$\lim_{|\overline{\Phi}_i\rangle\to|\Phi_a\rangle}\langle\Phi_b|e^{iH_0t}e^{-iHt}|\overline{\Psi_i^{(+)}}\rangle=e^{i(E_b-E_a)t}\langle\Phi_b|\Psi_a^{(+)}\rangle \qquad (117)$$

In an exactly analogous manner

$$\lim_{|\overline{\Phi}_i\rangle\to|\Phi_a\rangle}\langle\overline{\Phi}_b|e^{iH_0t}H_I e^{-iHt}|\overline{\Psi_i^{(+)}}\rangle=e^{i(E_b-E_a)t}\langle\Phi_b|H_I|\Psi_a^{(+)}\rangle \qquad (118)$$

It hence follows (using the facts that $\lim_{t_0\to-\infty}|\overline{\beta_i(t_0)}\rangle=0$ and $\lim_{t_0\to-\infty}|\overline{\rho_i^{(+)}(t_0)}\rangle=0$) that we have

$$w_{ba}=2\,\mathrm{Re}\{-i\langle\Phi_b|H_I|\Psi_a^{(+)}\rangle\langle\Phi_b|\Psi_a^{(+)}\rangle^*\} \qquad (119)$$

which we note to be time independent.

Using the Lippmann-Schwinger equation (102) we obtain

$$\langle \Phi_b | \Psi_a^{(+)} \rangle = \delta(b-a) + \left\{ P \frac{1}{E_a - E_b} - i\pi\,\delta(E_a - E_b) \right\} \langle \Phi_b | H_I | \Psi_a^{(+)} \rangle \tag{120}$$

and substitution into equation (119) gives

$$w_{ba} = 2\delta(b-a)\,\mathrm{Im}\,J_{ba} + 2\pi\,\delta(E_a - E_b)\,|J_{ba}|^2 \tag{121}$$

where $$J_{ba} = \langle \Phi_b | H_I | \Psi_a^{(+)} \rangle \tag{122}$$

This is the same expression as was obtained for the transition rate in the previous treatments, when we excluded the case $b = a$. In the present treatment this case is included in the first term in equation (121).

The above methods may also be employed to show the equivalence of time independent and time dependent definitions of the S -matrix. These may be taken respectively as

$$S_{ba} = \langle \Psi_b^{(-)} | \Psi_a^{(+)} \rangle \tag{123}$$

and $$S_{ba} = \lim_{|\Phi_i\rangle \to |\Phi_a\rangle} \lim_{t \to \infty} \lim_{t_0 \to -\infty} \langle \Phi_b | U(t, t_0) | \Phi_i \rangle \tag{124}$$

and it may be shown quite simply by the methods outlined above that both expressions reduce to

$$S_{ba} = \delta(b-a) - 2\pi i\,\delta(E_a - E_b)\,J_{ba} \tag{125}$$

the familiar expression previously obtained.

The proof is independent of the existence of bound states, provided they do not overlap the continuum, i.e. provided condition (1) (iii) is satisfied.

A treatment of the wave-packet approach to scattering theory has also been given by Sunakawa[11] who, by considering the precise form of the packet, succeeded in deriving the Lippmann-Schwinger equations, and the usual S -matrix expressions.

1.9 Characterisation of multi-channel scattering processes.

The theory as developed so far covers explicitly only the case of single-channel scattering where we have two colliding particles or, slightly more generally, several particles all infinitely separated both before and after collision. We shall now discuss what is meant by multi-channel scattering, show how the above theory is inadequate to describe it, and then set about constructing a substitute for the above reasoning. This work is due to Ekstein.[13]

The most characteristic feature of a scattering process is the occurrence of a finite number of non-interacting fragments at $t = -\infty$, and another set at $t = +\infty$. In finite times the fragments are in a finite region of space and undergo the interaction which produces the final set from the initial set. Each fragment for $t = \pm\infty$ is assumed to consist of a bound set of fundamental particles and the channel is determined by the constitution of these fragments, or, to be more precise, by the vanishing of the interaction between certain of the fundamental particles. For instance, if we have two particles, with position vectors $\underline{r}_1$ and $\underline{r}_2$ and an external potential which vanishes outside a region near the origin, then we can distinguish the four channels:-

(i) $|\underline{r}_1|$, $|\underline{r}_2|$, $|\underline{r}_1 - \underline{r}_2|$ large

(ii) $|\underline{r}_1|$ small; $|\underline{r}_2|$, $|\underline{r}_1 - \underline{r}_2|$ large

(iii) $|\underline{r}_2|$ small; $|\underline{r}_1|$. $|\underline{r}_1 - \underline{r}_2|$ large

(iv) $|\underline{r}_1 - \underline{r}_2|$ small; $|\underline{r}_1|$, $|\underline{r}_2|$ large

(i) describes the channel in which both particles are free.

(ii) and (iii) are the cases for which one particle is bound to the external potential and the other particle is free.

(iv) is the channel for which the two particles form a bound fragment, but outwith the range of the external potential.

The case in which both particles are bound by the external potential is excluded since this does not describe a scattering state.

1.10 The inadequacy of previous theory for multi-channel processes.

In our original theory for single-channel scattering, we transformed to the interaction picture:

$$|\Psi'(t)\rangle = e^{iH_0t}|\Psi(t)\rangle \tag{3}$$

If $|\Psi(t)\rangle$ is expressed as a wave packet and the interaction is of limited range, then it can be shown for a wide class of wave packets that as $|t| \to \infty$ then $|\Psi(t)\rangle$ is outside the range of the interaction, and hence its further development will be described by the equation

$$H_0|\Psi(t)\rangle = i\frac{\partial}{\partial t}|\Psi(t)\rangle \tag{126}$$

and so

$$|\Psi(t)\rangle = e^{-iH_0t}|\Psi'_\pm\rangle \tag{127}$$

as $t \to \pm\infty$

This means, of course, that our interaction picture state vector, given by equation (9) above, is assured of the limits

$$|\Psi'(\pm\infty)\rangle = |\Psi'_\pm\rangle \tag{128}$$

and so it is reasonable to assume the existence of an operator S defined by

$$|\Psi'_+\rangle = S|\Psi'_-\rangle \tag{129}$$

and that it is the limit of the operator $U(t, t_0)$ describing the development of the state vector in the interaction picture.

In the multi-channel case, however, we come across the difficulty that it is impossible to define a unique interaction picture which will ensure that the state vectors in that picture have a limit both for $t \to +\infty$ and for $t \to -\infty$ Let us illustrate this with reference to our above example. The total Hamiltonian is, using an obvious notation, of the form

$$H = H_0^{(1)} + H_0^{(2)} + V_1 + V_2 + V_{12} \tag{130}$$

If then in the far past we are in channel (ii) and in the distant future in channel (iii) (we say that the entrance channel is channel (ii) and the exit channel is channel (iii)) then in the far past the packet will be outwith the range of V_2 and V_{12}, whereas in the distant future the packet will be outside the range of V_1 and V_{12}. This means that if we define an interaction picture by

$$|\Psi'(t)\rangle = e^{i(H_0^{(1)} + H_0^{(2)} + V_1)t}\, |\Psi(t)\rangle \tag{131}$$

these state vectors will be assured of a limit as $t \to -\infty$ but will not have a limit for $t \to +\infty$. Similarly the interaction picture vector defined by

$$|\Psi'(t)\rangle = e^{i(H_0^{(1)} + H_0^{(2)} + V_2)t}\, |\Psi(t)\rangle \tag{132}$$

will have a limit for $t \to +\infty$ but not for $t \to -\infty$. Hence we see the impossibility of defining an operator S in accordance with equation (129) above, and so all our theory as developed above breaks down for the multi-channel case.

1.11 The development of a generalised S-matrix.

To provide an adequate replacement, the procedure suggested by Ekstein is as follows:-

Configuration space is divided into external and internal regions. In the external region at least one interaction term of the total Hamiltonian H (which is considered as containing a number of potential terms, time independent in the Schrödinger picture) vanishes, so that the system is described in the external region by a time independent solution of the Schrödinger equation which is a superposition of 'basis functions' $|\Phi_\alpha\rangle$, each basis function representing bound fragments not interacting with each other. Each channel is characterised by the vanishing of a potential term or terms, which we denote collectively by V_α. The operator

$$H - V_\alpha \equiv H_\alpha$$

may be regarded as made up of the kinetic energies of the mass centres of the various fragments, together with the Hamiltonian describing the internal structure of the fragments The basis functions are then products of plane wave functions for the mass-centres and the bound state eigenfunctions of the fragments, if all particles are distinguishable. For the case of indistinguishable particles we need to consider symmetrised or anti-symmetrised linear combinations in the usual way. In this case the effective Hamiltonians H_α belonging to a certain set of channels (e.g. (ii) and (iii) in the example given above) are connected by permutation operations; then we say that the basis function $|\Phi_a\rangle$ belongs to a group of channels. It is important to notice that the basis functions of one group of channels are neither mutually orthogonal, nor orthogonal to those of other channels. Due to the fact that the basis functions contain only bound states for the fragments, the set of basis functions for any one channel will not form a complete set (with the obvious exception of the 'free' channel in which every elementary particle has no interaction).

If bound states of the total system do not exist, then an argument similar to that given above for the single channel case will show that the wave packet $|\Psi(t)\rangle$ describing the system will asymptotically be in some part of the external region. Then the part of $|\Psi(t)\rangle$ which describes the channel α will obey the equation

$$i \frac{\partial}{\partial t}|\Psi(t)\rangle = H_\alpha |\Psi(t)\rangle \tag{133}$$

and so it is clear, using the definition of the basis functions given above that asymptotically we must have

$$|\Psi(t)\rangle = \int C_{\pm}(a)\, e^{-iE_a t}\, |\Phi_a\rangle\, da \qquad t \to \pm\infty \tag{134}$$

where the E_a are the eigenvalues of the various effective Hamiltonians for the different channels, and the integration also includes a summation over channels.

Now the basis functions are not orthonormal, but we may assume them to be normalised such that

$$\langle \Phi_b | \Phi_a \rangle = \delta(b-a) + g(b,a) \tag{135}$$

where $g(b,a)$ is square-integrable and bounded. This property leads to what is known as asymptotic orthogonality:-

$$\lim_{t\to\pm\infty} \langle \Phi_b | \Psi(t)\rangle = \lim_{t\to\pm\infty} \langle \Phi_b | \int C_{\pm}(a)\, e^{-iE_a t}\, |\Phi_a\rangle\, da$$

$$= C_{\pm}(b)\, e^{-iE_b t} + \lim_{t\to\pm\infty} \int C_{\pm}(a)\, g(b,a)\, e^{-iE_a t}\, da$$

using equation (135)

$$= C_{\pm}(b)\, e^{-iE_b t} \tag{136}$$

using the Riemann-Lebesgue lemma.

Now $|\langle\Phi_a|\Psi(t)\rangle|^2$ is, as usual, the probability of finding the system in the appropriate channel with certain momenta and bound states, and the asymptotic orthogonality obviously leads immediately to the fact that the asymptotic probabilities must be $|c_\pm(a)|^2$

Now, equation (134) yields, on the substitution of $e^{-iHt}|\Psi(0)\rangle$ for $|\Psi(t)\rangle$, the relationship

$$|\Psi(0)\rangle = \lim_{t\to\pm\infty}\int e^{i(H-E_a)t}c_\pm(a)|\Phi_a\rangle da \tag{137}$$

If we now apply equation (60), used by Gell-Mann and Goldberger to define a substitute limit, but here used merely as a mathematical tool, viz.

$$\lim_{t\to\pm\infty} f(t) = \pm\lim_{\epsilon\to 0}\epsilon\int_0^{\pm\infty} e^{\mp\epsilon t}f(t)dt \tag{138}$$

if the limit exists.

Writing $f(t)=e^{iAt}g$ with A Hermitian and g time independent leads to

$$\lim_{t\to\pm\infty} e^{iAt}g = \lim_{\epsilon\to 0}\left\{1-\frac{1}{A\pm i\epsilon}A\right\}g$$

Hence equation (137) becomes

$$|\Psi(0)\rangle = \lim_{\epsilon \to 0} \int C_{\pm}(a)\left\{1 - \frac{1}{H - E_a \pm i\epsilon}(H - E_a)\right\}|\Phi_a\rangle da$$

$$= \int C_{\pm}(a)\,|\Psi_a^{(\mp)}\rangle\, da \tag{139}$$

the last equation defining $|\Psi_a^{(\pm)}\rangle$; $|\Psi_a^{(\pm)}\rangle$ obviously satisfy

$$H\,|\Psi_a^{(\pm)}\rangle = E_a\,|\Psi_a^{(\pm)}\rangle \tag{140}$$

and are the outgoing and incoming solutions respectively, corresponding to the "plane wave" $|\Phi_a\rangle$.

It is now quite a simple matter, using the asymptotic orthogonality of the $|\Phi_a\rangle$ to prove that $|\Psi_a^{(+)}\rangle$ and $|\Psi_a^{(-)}\rangle$ each form an orthogonal set in the following sense:-

$$\int c(b)\,\langle \Psi_a^{(\pm)}|\Psi_b^{(\pm)}\rangle\, db = c(a) \tag{141}$$

for any square integrable $c(b)$.

Now we already have the result that

$$|\Psi(0)\rangle = \int C_{-}(a)\,|\Psi_a^{(+)}\rangle da = \int C_{+}(a)\,|\Psi_a^{(-)}\rangle da \tag{139}$$

and the S -matrix is defined as the integral operator connecting the coefficients of the initial packet $C_{-}(a)$ to those of the final packet $C_{+}(a)$. So, using equations (139) and (141) we obtain

$$c_+(b) = \int \langle \Psi_b^{(-)} | \Psi_a^{(+)} \rangle c_-(a) \, da \tag{142}$$

and hence $$S(b,a) = \langle \Psi_b^{(-)} | \Psi_a^{(+)} \rangle \tag{143}$$

(which is formally equivalent to a result in the single-channel case).

Using the definition of $|\Psi_a^{(\pm)}\rangle$ in equation (139) we obtain easily the equivalent forms:-

$$S(b,a) = \delta(b-a) - 2\pi i \, \delta(E_b - E_a) \langle (H-E_b)\Phi_b | \Psi_a^{(+)} \rangle \tag{144}$$

$$S(b,a) = \delta(b-a) - 2\pi i \, \delta(E_b - E_a) \langle \Psi_b^{(-)} | (H-E_a)\Phi_a \rangle \tag{145}$$

It is interesting to note that equations (144) and (145) reduce to the familiar single channel expressions if we write

$H = H_0 + H_I$ and use $H_0 |\Phi_a\rangle = E_a |\Phi_a\rangle$

The unitarity of the S -matrix is easily shown:-

$S^{-1}(b,a)$ is obviously defined by

$$c_-(b) = \int S^{-1}(b,a) c_+(a) \, da \tag{146}$$

and the orthogonality of the $|\Psi_a^{(\pm)}\rangle$ shows that

$$S^{-1}(b,a) = \langle \Psi_b^{(+)} | \Psi_a^{(-)} \rangle \tag{147}$$

Hence we have $S^*(b,a) = S^{-1}(a,b)$ and this shows the unitarity of the S -matrix for the general case.

Ekstein shows that, due to the fact that the complete set of basis functions is not linearly independent, they cannot be considered as eigenfunctions of an Hermitian operator H_0 ; he further shows that we can neither define an S -operator in the usual sense:-

$$|\Psi(t)\rangle = e^{-iH_0 t}\,|\Psi_\pm\rangle \quad , \quad t \to \pm\infty$$

$$\text{and} \qquad |\Psi_+\rangle = S\,|\Psi_-\rangle \tag{148}$$

nor can we define Møller wave operators by

$$|\Psi_a^{(\pm)}\rangle = \Omega_\pm\,|\Phi_a\rangle \tag{149}$$

We are hence forced to conclude that the matrix $S(b,a)$ must be regarded merely as an array of numbers rather than, as in the single-channel case, the matrix elements of a particular realisation of a linear operator defined in Hilbert space.

1.12 The relation of S-matrix elements to the cross-section.

We shall now show that the S -matrix as defined above is related to cross-sections in the same way as for the single-channel case, derived above in equation (50). It is important to note that strictly speaking the cross-section is defined only for the case where in the initial state we have two particles. The transition rate from which the cross-section is derived is of course valid in the general case. The coefficients $C_-(a)$ introduced above are known in the following manner:-

It is known that the wave packet is of the form

$$|\Psi(t)\rangle = \int c(a)\,|\Phi_a\rangle\, e^{-iE_a t}\, da \tag{150}$$

when it is sufficiently far from the target or scattering centre. Suppose then, we remove the target and obtain information on the beam at some time, say $t = 0$. This will give information concerning $c(a)$ since we now have (with the target removed) the beam described by equation (150) for all times. In particular

$$|\Psi(0)\rangle = \int c(a)\,|\Phi_a\rangle\, da \tag{151}$$

Replacing the target we now know that for this case the $c(a)$ will be the $C_-(a)$ for the scattering case.

The usual state of affairs under consideration is that, with the target removed, the state is characterised by the internal coordinates having sharp discrete quantum numbers a_0 while the momentum $\underline{k}$ is almost sharp. For definiteness we shall suppose the wave function has the form of a plane wave with propagation vector $k_z = K_0$, $k_x = k_y = 0$ inside a large box of side L and vanishes outside. This means that we will have

$$c_-(\underline{k},a) = \left(\frac{2}{\pi L}\right)^{3/2} \frac{\sin\frac{Lk_x}{2}}{k_x} \frac{\sin\frac{Lk_y}{2}}{k_y} \frac{\sin\frac{L(k_z-K_0)}{2}}{k_z-K_0}\delta_{aa_0} \qquad (152)$$

which is seen to give the usual box normalisation, and to lead to the required δ -functions as $L\to\infty$.

In fact, writing

$$\delta_L(x) = \frac{1}{\pi}\frac{\sin\frac{Lx}{2}}{x} \quad , \Delta_L(x) = \left(\frac{2}{\pi L}\right)^{\frac{1}{2}}\frac{\sin\frac{Lx}{2}}{x} \qquad (153)$$

we have

$$\lim_{L\to\infty} \delta_L(x) = \delta(x)$$

$$\lim_{L\to\infty} \{\Delta_L(x)\}^2 = \delta(x) \qquad (154)$$

Now, the asymptotic probability density for finding the system in the distant future in the state characterised by momentum $\underline{k}$ and discrete quantum numbers b is given by

$$P(\underline{k},b) = \left|\sum_{a'}\int\langle\underline{k},b|S|\underline{k}',a'\rangle\, c_-(\underline{k}',a')\, d^3\underline{k}'\right|^2 \qquad (155)$$

We must not let $L\to\infty$ at this stage since this would immediately yield the result that no scattering took place at all. This is due to the fact that then there would be an equal probability for finding the projectile at any point in space, and so its probability of hitting a finite target would be zero.

Suppose $N(\underline{k},b)$ be the number of measurements per unit time and that the current density of the projectiles is j . So the number of incident particles per unit time is $L^2 j$ and hence the number of measurements per unit time is given by

$$N(\underline{k},b) = L^2 j\, P(\underline{k},b) \tag{156}$$

$$= j\left|\sum_{a'}\int L\langle \underline{k},b|S|\underline{k}'a'\rangle c_-(\underline{k}',a')\,d^3\underline{k}'\right|^2 \tag{157}$$

and it is now that we must let $L\to\infty$, keeping j fixed.

Using equation (152) we obtain

$$N(\underline{k},b) = j\lim_{L\to\infty}\left|2\pi\int d^3\underline{k}'\langle \underline{k},b|S|\underline{k}'a_0\rangle\,\delta_L(k_x')\,\delta_L(k_y')\,\Delta_L(k_z'-K_0)\right|^2 \tag{158}$$

We now let $L\to\infty$ in $\delta_L(k_x')$ and $\delta_L(k_y')$ and apply equation (144) or (145), viz. $S(b,a) = \delta(b-a) - 2\pi i\,\delta(E_b-E_a)\,\mathcal{T}(ba)$ Then if $\underline{k}'\neq\underline{k}$ we get, for the integral,

$$-2\pi i\int dk_z'\langle \underline{k}\; b|\mathcal{T}|k_z' a_0\rangle\,\Delta_L(k_z'-K_0)\,\delta(E_b-E_a)$$

$$= -2\pi i\int\frac{dk_z'}{dE_a}\langle \underline{k},b|\mathcal{T}|k_z' a_0\rangle\,\Delta_L(k_z'-K_0)\,\delta(E_b-E_a)\,dE_a$$

$$= -2\pi i\,\langle \underline{k},b|\mathcal{T}|p_0 a_0\rangle\,\Delta_L(p_0-K_0)\,\frac{dp}{dE_a} \tag{159}$$

where E_a is the energy of the initial state as a function of the momentum, and p_0 is that momentum which will satisfy $E_{final} = E_a(p)$ (necessitated by the δ -function).

So we now have

$$N(\underline{k}, b) = \lim_{L\to\infty} j(2\pi)^4 \left| \langle \underline{k} b | J | p_0 a_0 \rangle \right|^2 \Delta_L^2(p_0 - K_0) \left(\frac{dp}{dE_a}\right)^2$$

$$= j(2\pi)^4 \left| J(b,a) \right|^2 \left(\frac{dp}{dE_a}\right)^2 \delta(p_0 - K_0)$$

$$= j(2\pi)^4 \left| J(b,a) \right|^2 \left(\frac{dp}{dE_a}\right)^2 \frac{dE_a}{dp} \delta(E_b - E_a) \tag{160}$$

Now, the differential cross-section is given by

$$d\sigma = d\Omega \int \frac{N k^2 dk}{j} \tag{161}$$

$$= d\Omega (2\pi)^4 \int \left| J(b,a) \right|^2 \frac{dp}{dE_a} k^2 \frac{dk}{dE_b} \delta(E_b - E_a)\, dE_b$$

$$= d\Omega (2\pi)^4 \left| J(b,a) \right|^2 \frac{dp}{dE_a} \frac{dk}{dE_b} k_b^2 \tag{162}$$

where all factors have to be evaluated for $E_b = E_a$. Using the relationships $\frac{dp}{dE_a} = V_a$, $\frac{dp}{dE_b} = V_b$ we obtain the familiar form for the cross-section

$$d\sigma = (2\pi)^4 \left| J(b,a) \right|^2 \frac{k_b^2}{V_a V_b} d\Omega \tag{163}$$

where, of course, $V_a = V_b$.

This means that we have shown that the same formal expression results from our multi-channel approach to theory, at least with the special form for $c(\underline{k}, a)$ as chosen here. A more general form may be handled by taking linear combinations of forms of the above type.

1.13 Conclusion

This ends our survey of formal scattering theory. We have seen that it shows us how to calculate cross-sections theoretically, which is what any scattering theory sets out to do, but it is at once obvious from the above discussions, with particular reference to the various limiting processes employed, that the whole theory is riddled with mathematical inconsistencies. As well, of course, we run into difficulties when we try to apply the above approach to quantum field theory. Quite apart from doubts concerning the validity of the interaction picture, we have the difficulty of the separation of the Hamiltonian such that the conditions (1) hold. In particular it has been shown for several simple cases that the eigenstates of H cannot be expressed as a linear combination of those of H_0, and hence even the argument due to Hack must break down. In the following chapter we set about trying to eliminate some of the mathematical inconsistencies of the theory.

CHAPTER II

RIGOROUS SCATTERING THEORY

2.1 Introduction and mathematical preliminaries.

In two papers Jauch[(7), (8)] reformulated scattering theory on a rigorous mathematical basis, and we shall now present a survey of this new approach to the subject. It is felt that much of the unsatisfactory nature of the previous theory, and, in particular, the impossibility of providing any mathematically meaningful definition of the S -matrix is due to the fact that the scattering states were represented by non-normalisable wave functions. This approach will now be replaced by that in which the state-vectors of a physical system are in one to one correspondence with the normalised elements of a Hilbert space, $\mathcal{H}$, the particular realisation of which is unimportant for much of the theory. The mathematical treatment is also kept meaningful by the avoidance of the use of any improper functions such as the Dirac δ -function, and the mathematical properties of Hilbert space circumvent the need for the introduction of these functions.

A Hilbert space consists of a set of elements $f, g, \cdots$ with the properties:-

(i) they form a linear vector space with respect to the complex numbers.

(ii) for any two elements f, g there exists a positive definite scalar product, denoted by (f, g) .

(iii) the space is complete.

(iv) the space is separable.

The distance between two elements f and g is defined as

$$\|f-g\| = \sqrt{((f-g),(f-g))} \tag{1}$$

The following two properties can be easily proved*

$$|(f,g)| \leq \|f\| \, \|g\| \tag{2}$$

$$\|f+g\| \leq \|f\| + \|g\| \tag{3}$$

Conditions (iii) and (iv) need further explanation. By the completeness of the space we mean the following:-

A sequence of elements $\{f_n\}$ belonging to $\mathcal{H}$ is said to be a fundamental sequence if $\|f_n - f_m\| < \epsilon$ for any $\epsilon > 0$ and all m, n greater than some N. The sequence is said to have a strong limit f if for any $\epsilon > 0$ there exists an N such that $\|f_n - f\| < \epsilon$ for all $n > N$. Then the space is said to be complete if every fundamental sequence has a strong limit.

(The concept of weak limit also needs to be introduced:

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* The proofs of these and other properties of Hilbert space may be found in many references. Particularly useful are (14), (15) and (16).

the sequence $\{f_n\}$ has the weak limit f if, for any g in $\mathfrak{H}$ and any $\epsilon > 0$ then there exists an N such that for all $n > N$, $|(g,f_n) - (g,f)| < \epsilon$).

The separability requirement means that there exists a sequence $\{f_n\}$ of elements of $\mathfrak{H}$ which is dense in $\mathfrak{H}$; i.e. given ϵ, $N > 0$ and f in $\mathfrak{H}$, then there exists an $n > N$ such that $\|f_n - f\| < \epsilon$ (every element of $\mathfrak{H}$ is a limit point of $\{f_n\}$).

An important result which is used is the spectral theorem, which is satisfied by self-adjoint operators In general any hermitian operator with dense domain and no proper extensions is self-adjoint. The theorem states that an operator A is self-adjoint if there exists a family of non-decreasing projection operators $E(\lambda)$ (i.e. if $\lambda_1 \leq \lambda_2$ $(f, E(\lambda_1)f) \leq (f, E(\lambda_2)f)$ for any f in $\mathfrak{H}$) such that $E(-\infty) = 0$, $E(\infty) = I$ (the unit operator)

and $$A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda) \tag{4}$$

in the sense that $(f, Ag) = \int_{-\infty}^{\infty} \lambda \, d(f, E(\lambda)g)$ where f and g are elements of $\mathfrak{H}$.

Now if $F(\lambda)$ is a function of the real variable λ we define the function $F(A)$ as

$$F(A) = \int_{-\infty}^{\infty} F(\lambda) \, dE(\lambda) \tag{5}$$

We shall often be dealing with a unitary operator of the form

$$U_t = e^{iAt}$$

where A is self-adjoint, and this means that we shall have

$$U_t = \int_{-\infty}^{\infty} e^{i\lambda t} \, dE(\lambda) \tag{6}$$

2.2 Scattering systems.

Let us now see how we apply these ideas to our scattering problem. The properties of any quantum mechanical system are described by means of a self-adjoint linear operator (the Hamiltonian) which operates in a Hilbert space $\mathcal{H}$. The region of $\mathcal{H}$ in which H is defined is called the domain of H and is denoted by D_H ; it is everywhere dense in $\mathcal{H}$. In general H is unbounded, but has, however, a lower bound:

$$(f, Hf) \geq 0 \qquad \text{for all } f \text{ in } D_H \tag{7}$$

It is possible to fix the lower bound arbitrarily as zero since we can always introduce an additive constant into the definition of H . So that we may discuss scattering, there are certain restrictions that we must make on H in order that it should describe what we shall call 'simple scattering systems'; the number of particles taking part in the process being restricted to one or two, and being the same both before and after the collision.

Now, as we have mentioned before, the characteristic feature of a scattering process is that both in the distant past and future we have essentially free or non-interacting particles which are described by a Hamiltonian H_0, the energy operator of the particles under consideration, the particular form of which depends on whether we treat the particles relativistically or not. In the case of a simple scattering system H_0 will be the same both before and after the collision, whereas in the more complicated multi-channel case, when the number of fragments is not conserved, H_0 will, in general, have a different form in the distant future as compared with the far past.

So, considering the single-channel case we have the self-adjoint linear operator H_0 which also has a lower bound:

$$(f, H_0 f) \geq 0 \quad \text{for all } f \text{ in } D_{H_0} \tag{8}$$

We now introduce the unitary operators U_t and V_t defined by

$$U_t = e^{-iH_0 t} \tag{9}$$

$$V_t = e^{-iHt} \tag{10}$$

The unitary property is seen to be expressed by

$$U_t^{\dagger} = U_{-t} \tag{11}$$

$$V_t^{\dagger} = V_{-t} \tag{12}$$

The great advantage of introducing these operators is that, because of their unitarity, they are defined everywhere in $\mathcal{H}$ whereas the operators H and H_0 are defined only in the regions D_H and D_{H_0} respectively.

Jauch now postulates three conditions a scattering system must satisfy:-

I The limits

$$\lim_{t \to \mp\infty} V_t^{\dagger} U_t f = f_{\pm}$$

exist for all f in $\mathcal{H}$.

II If the sets of elements f_+ and f_- are denoted by R_+ and R_- respectively, then we must have $R_+ = R_-$ ($= R$, say).

III Define the subspace M to be that which is spanned by the eigenvalues of H , i.e. all elements which satisfy

$$Hf = \omega f \qquad \text{with } \omega \text{ real.}$$

If then we denote the orthogonal complement of M by N (elements in N are said to belong to the continuum of H)

we require that $N \subset R$ (where the symbol '$\subset$' means 'is contained in'). It can be shown that $R \subset N$ always. Hence this condition is equivalent to $R = N$.

Let us now see what these conditions mean physically. The elements $f_\pm$ correspond to what in our previous treatment we called outgoing and incoming scattered waves respectively. Now it is easy to show that if the limits in I exist, then we also shall have

$$\lim_{t \to \mp\infty} U_t^\dagger V_t f_\pm = f \tag{13}$$

Now, the element $g(t) = U_t^\dagger V_t g$ merely represents the state of a system in the interaction picture when the element g represents it in the Heisenberg picture. So what we are asserting by the existance of the above limits is that the states in the interaction picture approach a constant in the distant past or future, and, since any change is due only to the interaction operator $H - H_0$, this means that asymptotically the interaction must become ineffective. Condition II obviously puts past and future on the same footing.

What has just been said applies equally well to simple and multi-channel scattering if we consider suitable definitions of H_0. What condition III does is to characterise the simple scattering system. It says that there are no states in the continuum of H which are not scattering states whose asymptotic behaviour is determined by the Hamiltonian H_0. If there were such states then it is obvious that the description of the scattering states in terms of H and H_0 would be incomplete, and it is just this state of affairs that characterises multi-channel scattering. We hence adopt condition III as that postulate necessary to characterise simple scattering systems.

2.3 The scattering operator.

Now it is obvious that the mapping of f in $\mathcal{H}$ on to f_+ or f_- in R is a linear mapping, and it can easily be shown that it is also an isometry: i.e. $\|f\| = \|f_\pm\|$ It hence follows that we can define bounded linear operators $\Omega_\pm$ such that

$$\Omega_\pm f = f_\pm \tag{14}$$

and the bound of these operators will be unity. (Here we should note that the definition of the bound of an operator A is given by

$$\|A\| = \text{least upper bound of } \frac{(f, Ag)}{\|f\| \; \|g\|} \tag{15}$$

where f and g are any two elements of $\mathcal{H}$).
These operators are called the wave operators, and may easily

be shown to have the following properties:-

If Q is the set of elements f such that $\Omega^{\dagger}_{\pm} f = 0$ then Q is identical with the orthogonal complement of R, $R^{\perp}$, and it follows that

$$\text{(i)} \quad \Omega_{\pm} \Omega^{\dagger}_{\pm} = E_R = I - E_Q \tag{16}$$

$$\text{(ii)} \quad \Omega^{\dagger}_{\pm} \Omega_{\pm} = I \tag{17}$$

where E_R and E_Q are the projection operators on to the sets R and Q respectively and I is the unit operator

$$\text{(iii)} \quad \Omega^{\dagger}_{\pm} E_Q = E_Q \Omega_{\pm} = 0 \tag{18}$$

Hence if we define an operator S (the scattering operator) by the relation

$$S = \Omega^{\dagger}_{-} \Omega_{+} \tag{19}$$

it follows immediately from the above results that the unitarity of this operator will be ensured. It is important to note, however, that the steps of the argument leading to this unitarity are valid only if we have $R_+ = R_-$. Thus our condition II is necessary in order to ensure the unitarity of the scattering operator.

2.4 The integral representation of the wave operators.

In the previous chapter integral representations of the wave operators were given which were, however, purely formal in character: in particular, no proof of their existence was given. Jauch gives an existence proof for integral representations which is, however, too long to be reproduced in full here. What we shall do is to state the various lemmata proved by Jauch, giving an indication of the proof in any of the cases where there is some difficulty:-

Lemma 1.

The elements $U_t f$ and $V_t f$ are strongly continuous functions of the real variable t for all t and all f in $\mathfrak{H}$ (Strong continuity of $U_t f$ at $t=t_0$ means that given any $\epsilon > 0$ then there exists a $\delta > 0$ such that $\|U_t f - U_{t_0} f\| < \epsilon$ for $|t-t_0| < \delta$).

To prove this lemma we use the fact that continuity at $t=0$ is equivalent to continuity at $t=t_0$, and then use the spectral theorem (5), viz.

$$(f, U_t f) = \int_{-\infty}^{\infty} e^{-i\lambda t}\, d\,(f, E(\lambda) f) \tag{20}$$

to show that if f is in the domain of H_0 that

$$\|(1-U_t)f\| \leq t.C \tag{21}$$

where C is some fixed positive number.

Then, using the fact that the domain of H_0 is dense in $\mathcal{H}$ the result follows almost immediately.

Lemma 2.

If $W_t = V_t^+ U_t$ the function $W_t f$ is strongly continuous for all t and all f in $\mathcal{H}$.

Lemma 3.

The function $\phi(t) \equiv (f, W_t g)$ is continuous for all t and all f, g in $\mathcal{H}$. Also $|\phi(t)| \leq \|f\| \, \|g\|$.

Lemma 4.

The integral $\Phi_\epsilon(f,g) = \epsilon \int_0^\infty e^{-\epsilon t} \phi(t) \, dt$ is absolutely convergent for all f, g in $\mathcal{H}$ and $\epsilon > 0$, and it is bounded:

$$|\Phi_\epsilon(f,g)| \leq \|f\| \, \|g\|$$

Now, a function $\Psi(f,g)$ is called a bilinear functional if it satisfies the three properties

(i) $\Psi(\lambda_1 f_1 + \lambda_2 f_2, g) = \lambda_1^* \Psi(f_1, g) + \lambda_2^* \Psi(f_2, g)$

(ii) $\Psi(f, \lambda_1 g_1 + \lambda_2 g_2) = \lambda_1 \Psi(f, g_1) + \lambda_2 \Psi(f, g_2)$

(iii) $|\Psi(f,g)| \leq B \|f\| \, \|g\|$ (B fixed and > 0) (22)

and it can be shown that for every bilinear functional $\Psi(f,g)$ there exists a bounded linear operator A such that

$$\Phi(f,g) = (f, Ag) \tag{23}$$

So, from Lemma 4 it immediately follows that $\Phi_\epsilon(f,g)$ is a bilinear functional, and hence there must exist a bounded linear operator Ω_ϵ such that

$$\Phi_\epsilon(f,g) = (f, \Omega_\epsilon g)$$

$$\text{and} \quad \|\Omega_\epsilon\| = 1 \tag{24}$$

The operator which is thus defined may be written,

$$\Omega_{-\epsilon} = \epsilon \int_0^\infty e^{-\epsilon t} U_t^+ V_t \, dt \qquad (\epsilon > 0) \tag{25}$$

and we can similarly show the existence of a corresponding operator

$$\Omega_{+\epsilon} = \epsilon \int_{-\infty}^0 e^{\epsilon t} U_t^+ V_t \, dt \qquad (\epsilon > 0) \tag{26}$$

For properties common to both operators we shall merely talk about Ω_ϵ. What we want to do is study the limits of these operators as ϵ approaches zero. Now the limit of a set of operators may be understood in any one of three senses:-

(i) Limit in the norm: Ω_ϵ converges in the norm to Ω $(\Omega_\epsilon \Rightarrow \Omega)$ if $\|\Omega_\epsilon - \Omega\| \to 0$

(ii) The strong limit: Ω_ϵ converges strongly to Ω $(\Omega_\epsilon \to \Omega)$ on a subset M of $\mathcal{H}$ if, for all f in M,

(iii) The weak limit: Ω_ϵ converges weakly to Ω $(\Omega_\epsilon \rightharpoonup \Omega)$ on a subset M of $\mathcal{H}$ if, for any f and g in M

$$|(f, \Omega_\epsilon g) - (f, \Omega g)| \to 0$$

We shall consider limits as being in the strong sense, which imply the existence of weak limits.

Lemma 5.

The set L of elements f in $\mathcal{H}$ such that $\lim_{\epsilon \to 0} \Omega_\epsilon f$ exists is a closed linear manifold and $\lim_{\epsilon \to 0} \Omega_\epsilon$ is a bounded linear operator on L with bound ≤ 1 .

The fact that L is a linear manifold is obvious. That it is closed may be shown by taking a sequence f_n in L such that $f_n \to f$ and proving that f also is in L .

Lemma 6.

If $\lim_{t \to +\infty} V_t^+ U_t f = f_-$ exists, then

$$\lim_{\epsilon \to 0} \epsilon \int_0^\infty e^{-\epsilon t} V_t^+ U_t f \, dt$$ also exists, and is

equal to f_- .

The proof of this lemma is quite straightforward, and it leads to the interesting conclusion that for scattering systems

for which we have postulated the existence of $\lim_{t\to\pm\infty} V_t^+ U_t$ throughout $\mathcal{R}$ (condition I) then we must also have $\lim_{\epsilon\to 0} \Omega_\epsilon$ existing throughout $\mathcal{R}$ or, in other words, $L = \mathcal{R}$.
It hence follows that we may define

$$\Omega_\pm = \lim_{\epsilon\to 0} \Omega_{\pm\epsilon} \tag{27}$$

It can now be shown that the adjoint operators $\Omega_\pm^\dagger$ may be similarly defined by

$$\Omega_\pm^\dagger = \lim_{\epsilon\to 0} \Omega_{\pm\epsilon}^\dagger \tag{28}$$

This is not an immediate consequence of equation (27) as one might have expected; this is due to the fact that the existence of the limit in equation (28) follows from Lemma 6 only if the limit of the integrand exists, and this is true only if f is in R . Hence a separate and rather complicated proof is required, which will not be given here (although we might note that the proof requires that H_0 should have no point spectrum).

So we have now obtained the result that the operators

$$\Omega_- = \lim_{\epsilon\to 0} \epsilon \int_0^\infty e^{-\epsilon t} V_t^+ U_t \, dt \tag{29}$$

$$\Omega_-^+ = \lim_{\epsilon\to 0} \epsilon \int_0^\infty e^{-\epsilon t} U_t^+ V_t \, dt \tag{30}$$

exist throughout the Hilbert space $\mathcal{H}$, and are bounded linear operators with upper bound unity and satisfy the relations

$$\Omega_-^{\dagger}\Omega_- = I \qquad \Omega_-\Omega_-^{\dagger} = I - E_Q \tag{31}$$

Also, a similar result holds for the operators

$$\Omega_+ = \lim_{\epsilon\to 0} \epsilon\int_{-\infty}^{0} e^{\epsilon t} V_t^{\dagger} U_t \, dt \tag{32}$$

$$\Omega_+^{\dagger} = \lim_{\epsilon\to 0} \epsilon\int_{-\infty}^{0} e^{\epsilon t} U_t^{\dagger} V_t \, dt \tag{33}$$

It hence follows that we have arrived at the same integral representation of the wave operators as obtained by Gell-Mann and Goldberger, but now we have used only rigorous and meaningful mathematics, and deal with precisely defined quantities.

Using the above integral representations it is possible to derive further properties of the wave operators. These properties will merely be stated, the proofs being straightforward in most cases:-

(1) The operators $\Omega_{\pm}$ are intertwining operators with respect to the operators U_t and V_t , i.e.

$$\Omega_{\pm} U_t = V_t \Omega_{\pm} \tag{34}$$

Similarly $\Omega_{\pm}^{\dagger} V_t = U_t \Omega_{\pm}^{\dagger}$ (35)

(2) If we define $\Omega_\pm(t) = U_t^+ \Omega_\pm U_t$ (36)

then $\lim_{t\to\pm\infty} \Omega_\pm(t)$ exist and

(a) $\lim_{t\to-\infty} \Omega_+(t) = \lim_{t\to+\infty} \Omega_-(t) = I$ (37)

(b) $\lim_{t\to+\infty} \Omega_+(t) = S$ (38)

(c) $\lim_{t\to-\infty} \Omega_-(t) = S^{-1}$ (39)

This result provides us with a new and completely equivalent definition of the scattering operator.

2.5 The physical interpretation of the scattering operator.

The physical interpretation of the scattering operator is obtained as follows: initially suppose we are in the state f which, because there is no interaction will have its time dependence given by $U_t f$ for $t \to -\infty$. Then, because of the basic postulate I for scattering systems, this will be the same (in the norm) as $V_t f_+$ for $t \to -\infty$. This will, however, describe the state for all time, since V_t gives time dependence with the interaction included. So the probability of being in the state g in the distant future (which has time dependence given by $U_t g$ since there is no interaction) will be given by

$$\lim_{t\to+\infty} |(U_t g, V_t f_+)|^2 = \lim_{t\to+\infty} |(V_t^+ U_t g, f_+)|^2$$

$$= |(g_-, f_+)|^2$$

which obviously has the equivalent form

$$|(\Omega_- g, \Omega_+ f)|^2$$

$$= |(g, \Omega_-^+ \Omega_+ f)|^2$$

$$= |(g, Sf)|^2 \tag{40}$$

From this probability cross-sections may be calculated by giving a particular realisation to the elements of Hilbert space, and assuming an appropriate integral representation for S . The result finally arrived at is the same as that obtained in the previous chapter.

2.6 The validity of the basic postulates.

We have now succeeded in putting our theory on a mathematically satisfying basis (at least for single-channel scattering so far), but our work is by no means finished. We must examine the basic postulates for scattering systems, and find out whether they are satisfied for particular physical systems. So what we have to do is

(i) to examine the existence of the strong limits

$$\Omega_{\mp} = \lim_{t \to \mp\infty} V_t^+ U_t$$

(ii) to examine the validity of the assumption $R_+ = R_-$ (which, as we noted above, is equivalent to the unitarity of the S-operator).

The first rigorous mathematical investigation into (i) was made by Cook(17) prior to the work of Jauch which we have outlined above. He succeeded in proving the existence of the wave operators for scattering by a potential $V(\underline{x})$ satisfying the condition

$$\int_{-\infty}^{\infty} |V(\underline{x})|^2 d^3\underline{x} < \infty \tag{41}$$

A more general investigation has been made by Jauch and Zinnes(18) and we shall now present an outline of their paper.

We have the self-adjoint and (in general) unbounded operators H and H_0 (total and free Hamiltonians respectively). The meaning of the interaction operator H_I has to be made precise. It will be given by $H_I = H - H_0$ on the intersection of D_H and D_{H_0}. This will in general permit an extension either to all of $\mathcal{H}$ or to a linear manifold dense in $\mathcal{H}$. What we are going to do is to place a restriction on the interaction operator (confine ourselves to studying what we shall call 'admissible interaction operators') and then obtain an equivalent condition to postulate I . We then consider the case where the interaction is a central potential and derive the conditions for admissibility; it is then possible to specialise still further by assuming the potential to be of the form $1/r^{\beta}$ and we may then derive the range of values of β for which the postulate I holds good.

Admissible interaction operators are defined as follows:- an interaction operator H_I is admissible if there exists a linear manifold D'_{H_I} , dense in $\mathcal{H}$, on which H_I and H_0 are defined and which is left invariant under all operators of the group U_t . Any bounded interaction operator is admissible, but the admissible class also contains such unbounded interactions as the Coulomb or the Yukawa potentials.

Jauch and Zinnes show the equivalence of postulate I with the condition that the strong limit

$$\lim_{t \to \infty} U_t^{+} V_\tau U_t = U_\tau \tag{42}$$

should exist throughout $\mathcal{H}$ for all $\tau \geq 0$ (The proof is quite straightforward, utilising the identity

$$\|(V_t^{+} U_t - V_{t'}^{+} U_{t'}) f\| = \|(U_t^{+} V_\tau U_t - U_\tau) f\|$$

where $\tau = t - t'$).

We now define the expression

$$\Phi_t(f,g) \equiv (HV_t f, U_t g) - (V_t f, H_0 U_t g) \tag{43}$$

where f is in D_H and g is in D_{H_0}.

It follows quite simply that $\Phi_t(f,g)$ is a continuous function of t for all real t and any fixed pair of elements f , g . Hence $\Phi_t(f,g)$ can be integrated over any finite t -interval and we define

$$\Psi_{t_2 t_1}(f,g) \equiv \int_{t_1}^{t_2} \Phi_t(f,g)\,dt \tag{44}$$

Then using the property that

$$\lim_{\tau \to 0} \frac{V_{t+\tau} - V_t}{\tau} f = -iHV_t f \tag{45}$$

and a similar result for H_0 and U_t we obtain the result that

$$\Psi_{t_2 t_1}(f,g) = (f, (W_{t_2} - W_{t_1})g) \tag{46}$$

where
$$W_t = V_t^+ U_t \tag{47}$$

Now, we saw that the left hand side of equation (46) was defined only for f in D_H and g in D_{H_0} , whereas we now see that the right hand side, because of the unitarity of W_t , is defined for all f , g in $\mathcal{H}$.

Let us now assume that we are dealing with an admissible interaction operator, and that g is in D'_{H_I} which is, of course, contained in D_{H_0} . Then we shall have

$$(HV_t f, U_t g) = (f, V_t^+ H U_t g) \tag{48}$$

and hence

$$(f, (W_{t_2} - W_{t_1}) g) = i \int_{t_1}^{t_2} dt \, (f, V_t^+ H_I U_t g) \tag{49}$$

where the right hand side is defined for f in D_H and g in D'_{H_I} . Once again we see that the expression on the left hand side provides an extension for the expression on the right hand side.

It follows that the integral may be considered as a bilinear functional in all of $\mathcal{H}$ and hence leads to the definition of the bounded linear operator

$$X_{t_2 t_1} = i \int_{t_1}^{t_2} dt \, V_t^+ H_I U_t \tag{50}$$

It hence follows that we have the operator relation

$$W_{t_2} - W_{t_1} = i \int_{t_1}^{t_2} dt \, V_t^+ H_I U_t \tag{51}$$

and it can now be seen almost immediately by using the Cauchy convergence condition and the completeness of $\mathcal{H}$ that a necessary and sufficient condition for postulate I to be true is that for an admissible interaction operator H_I , the operator

$$X_{t_2 t_1} = i\int_{t_1}^{t_2} V_t^+ H_I U_t \, dt \tag{50}$$

should have a strong limit throughout $\mathcal{H}$ for $t_2 \to +\infty$ and $t_1 \to -\infty$

In order to make matters even more specific we consider scattering by a central potential $V(r)$ and we shall realise our Hilbert space by $L^2(E_3)$ (i.e. modulus squared Lebesgue integrable) functions $\psi(\underline{x})$ over three dimensional Euclidean space.

The total Hamiltonian is assumed to be of the form

$$H = \frac{p^2}{2m} + H_I \tag{52}$$

where $$(H_I\psi)(\underline{x}) = V(r)\psi(\underline{x}) \tag{53}$$

and, of course, $$(p^2\psi)(\underline{x}) = -\nabla^2\psi(\underline{x}) \tag{54}$$

We must now look at the admissibility of the interaction operator in the above sense, and this Jauch and Zinnes do by means of Wiener's Theorem[19] which says that there exist suitable generating functions which, when displaced and superposed can approximate in the norm any function in $L^2(E_3)$ to any desired accuracy:-

If $K_1(\underline{x})$ belongs to $L^2(E_3)$ and is such that its Fourier transform vanishes almost nowhere (i.e. only on a set of measure zero) then if $K_2(\underline{x})$ belongs to $L^2(E_3)$ and $\epsilon > 0$, there exists an integer N together with a set of real vectors $\underline{\Lambda}_n$ and complex numbers A_n such that

$$\int_{-\infty}^{\infty} \left| K_2(\underline{x}) - \sum_{n=1}^{N} A_n K_1(\underline{x} - \underline{\Lambda}_n) \right|^2 d^3\underline{x} < \epsilon$$

Now the set of functions $\sum_{n=1}^{N} A_n K_1(\underline{x} - \underline{\Lambda}_n) \equiv F(\underline{x})$

for all finite N and $\underline{\Lambda}_n$ form a linear manifold $\mathcal{L}$. So, according to Wiener's theorem, if $K_1(\underline{x})$ satisfies the required conditions, then $\mathcal{L}$ is dense in $\mathcal{H}$. This is useful for our present requirements for what we want to show is that under certain conditions on H_I there exists a linear manifold D'_{H_I} dense in $\mathcal{H}$ on which H_I and H_0 are defined, and which is invariant under U_t.

The conditions given by Jauch and Zinnes for this to be true is that there should exist positive R, M_1 and M_2 such that

$$\int_0^R r^2 V^2(r)\, dr < M_1 \qquad \int_R^{\infty} e^{-r^2} V^2(r)\, dr < M_2 \tag{55}$$

(These conditions are easily seen to include both Coulomb and Yukawa type potentials).

For the generating functions of the linear manifold we choose the Fourier transforms of

$$\phi_{\underline{k}_0}(\underline{k}) = e^{-a^2(\underline{k}-\underline{k}_0)^2} \quad \text{for real } a>0 \tag{56}$$

which are easily seen to satisfy the conditions of Wiener's Theorem. It can now be shown that, under the conditions (55), every function of the form $U_t\phi_{\underline{k}_0}$ with arbitrary t and $\underline{k}_0$ is in D_{H_0} as well as in D'_{H_I}. This must also be true for linear combinations of such functions and, by Wiener's Theorem, such functions generate the linear manifold $\mathcal{L}$, everywhere dense in $\mathfrak{H}$. So with $D'_{H_I} = \mathcal{L}$ we have obtained a linear manifold which satisfies the required conditions and we do in fact have an admissible interaction operator.

If we specialise still further and assume $V(r) = 1/r^\beta$ then it is obvious that conditions (55) are satisfied, and it can also be shown that the requirement that $X_{t_2t_1} = \int_{t_1}^{t_2} V_t^+ H_I U_t\,dt$ should converge is also satisfied if we have $1 < \beta < 3/2$. Unfortunately this method of approach has not shown that the Coulomb potential admits a scattering operator.

Kuroda[20] has also studied this problem using even more subtle mathematical techniques. He assumed the total Hamiltonian to be of the form

$$H = -\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} + V(\underline{x}) \qquad (57)$$

with $\underline{x} = (x_1 \cdots x_m)$

which is defined in E_m (m dimensional Euclidean space) and succeeded in showing the existence of the limits required in condition I under the restriction that there should exist a positive number ϵ such that

$$V(\underline{x})\,(1+r)^{-\left(\frac{m}{2}-1\right)+\epsilon} \quad \text{belongs to } L^2(E_m) \qquad (58)$$

$$(\, r = |\underline{x}| \,)$$

This result was also proved by Hack[21] for the case $m=3$ (It is important to note that for $m=3$ the condition that $V(\underline{x})$ belongs to $L^2(E_3)$ which was, of course, the condition obtained by Cook[17] will imply the above restriction, (58)).

Kuroda, in the same paper, also investigated problem (ii) (i.e. the equivalence of R_+ and R_-) and proved this equivalence under the restrictions for the potential that it should satisfy, in addition to condition (58), either

(a) $m=1$

or (b) $m \geqslant 2$ and $V(\underline{x}) = V(r)$

He also succeeded in giving another condition for this equivalence for the case $m \leq 3$. In this case the condition reduces to the fact that the potential must belong to both $L^1(E_m)$ and $L^2(E_m)$.

Kato and Kuroda[22] have given an example, as an illustration of the fact that the existence of the wave operators does not necessarily imply the unitarity of the scattering operator. They consider the total Hamiltonian

$$H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + V(x_2) + K \qquad (59)$$

describing two particles moving in E_1 (there being no essential difficulty in replacing E_1 by E_3). $V(x_2)$ belongs to both $L^1(E_1)$ and $L^2(E_1)$, and the interaction K is a self-adjoint operator such that, for any element f in $\mathcal{H}$ we have

$$Kf = c(f,\phi)\phi \qquad (60)$$

where ϕ is a fixed function in $\mathcal{H}$ and c is a real number.

Kato and Kuroda then show that the wave operators Ω_+ and Ω_- exist, but, however, that the scattering operator $\Omega_-^+\Omega_+$ is not unitary, or, as we have seen to be equivalent, R_+ is not identical with R_- .

2.7 The characterisation of multi-channel scattering.

Let us now return to the reformulation of multi-channel scattering as proposed by Jauch.[8] The characterisation of multi-channel processes was mentioned in the previous chapter when we gave a review of Ekstein's work. Jauch adopts a slightly different point of view. Instead of considering a channel as being specified by the vanishing of a potential term or terms we take the following approach: each channel is characterised by a certain set of free fragments occurring in the distant past or future. Each fragment is taken as consisting of a bound set of fundamental particles, and is characterised by the momentum operator P_i which is the sum of the momentum operators of the fundamental particles constituting the fragment. The free asymptotic motion of the fragments is then described (non-relativistically) by the self-adjoint Hamiltonian

$$H_\alpha = \sum_i \frac{1}{2m_i} P_i^2 \qquad (61)$$

where $\{m_i\}$ is the set of masses of the fragments or, (relavistically) by

$$H_\alpha = \sum_i \sqrt{m_i^2 + P_i^2} \qquad (62)$$

The various channels provide us with a set of channel Hamiltonians $\{H_\alpha\}$. It is convenient to introduce the 'channel operator' defined by

$$\mathcal{U}_t^{(\alpha)} \equiv e^{-iH_\alpha t} \tag{63}$$

These operators have the properties

(i) $[\mathcal{U}_t^{(\alpha)}, \mathcal{U}_{t'}^{(\beta)}] = 0$ for all α, β, t, t' (64)

(ii) $\mathcal{U}_t^{(\alpha)} f = \mathcal{U}_t^{(\beta)} f$ for all t implies

either $f = 0$ or $\alpha = \beta$ (65)

These channel operators are not usually known explicitly. What is known is the total Hamiltonian H , or the corresponding group of unitary operators $V_t = e^{-iHt}$

Now the different fragments describing different channels are described by the channel operators defined by equation (63). In order that a unitary group $\mathcal{U}_t^{(\alpha)}$ with the properties (64) and (65) should describe a channel of the system with total Hamiltonian H , we require that for at least one non-zero element in $\mathcal{K}$ the strong limits

$$\lim_{t \to \mp\infty} V_t^{\dagger} \mathcal{U}_t^{(\alpha)} f = f_{\pm}^{(\alpha)} \tag{66}$$

should exist. The physical interpretation of this requirement is obvious from our discussion of the single-channel case, and is clearly a reasonable one.

It may be shown that if there exists one element such that the limits (66) exist, then there exists an infinite dimensional closed linear manifold D_α throughout which the limits exist, and which is invariant under the operators of the group

Now, the mappings $f \to f_\pm^{(\alpha)}$ are linear isometries, similar to the corresponding single channel result. It follows that the ranges $R_\pm^{(\alpha)}$ of $f_\pm^{(\alpha)}$ must be closed linear manifolds. The sets of all subspaces $\{R_\pm^{(\alpha)}\}$ as α covers all the channels will span linear manifolds, the closures of which we denote by $R_\pm$. As before we denote by N the subspace of continuum states of H, i.e. the orthogonal complement of M, the subspace of proper elements of H. Then the generalisation of the second and third conditions in the single channel case is that we should have

$$R_+ = R_- = N \tag{67}$$

As before, we have bounded linear operators $\Omega_\pm^{(\alpha)}$ defined on D_α by

$$\Omega_\pm^{(\alpha)} f = f_\pm^{(\alpha)} \tag{68}$$

where f is in D_α

(This definition can always be extended so that $\Omega_{\pm}^{(\alpha)}$ will have as domain the whole of $\mathcal{H}$: if $f = g + h$ where g is in D_α and h is in $D_\alpha^{\perp}$, then we define $\Omega_{\pm}^{(\alpha)} f$ by the relationship

$$\Omega_{\pm}^{(\alpha)} f = \Omega_{\pm}^{(\alpha)} g \tag{69}$$

the right-hand side of which is well-defined since g is in D_α).

Zinnes[23] has proved a uniqueness theorem on the free Hamiltonians for a multi-channel scattering system:-

Let $\{U_t^{(\alpha)}\}$ and $\{\bar{U}_t^{(\alpha)}\}$ $(\alpha = 1, 2 \cdots)$ be two sets of unitary groups such that

$$U_t^{(\alpha)} = e^{-iH_\alpha t} \qquad \bar{U}_t^{(\alpha)} = e^{-i\bar{H}_\alpha t} \tag{70}$$

where
$$H_\alpha = \sum_{j=1}^{n} a_j^\alpha P_j^2 \qquad \bar{H}_\alpha = \sum_{j=1}^{n} \bar{a}_j^\alpha P_j^2 \tag{71}$$

and $\{a_j^\alpha\}$ $\{\bar{a}_j^\alpha\}$ are sets of real numbers. Also let V_t be a unitary group such that for each α

$$\Omega_{\pm}^{(\alpha)} = \lim_{t \to \mp\infty} V_t^{+} U_t^{(\alpha)} \tag{72}$$

$$\bar{\Omega}_{\pm}^{(\alpha)} = \lim_{t \to \mp\infty} V_t^{+} \bar{U}_t^{(\alpha)} \tag{73}$$

exist on closed domains D_α and $\bar{D}_\alpha$ with ranges R_α and $\bar{R}_\alpha$ respectively. Then if $R_+ = \bar{R}_+ = R_- = \bar{R}_-$ the sets H_α and $\bar{H}_\alpha$ are identical except possibly for the order of the elements.

The proof is too long to be given here, but we may note that it can be extended to include the case where the free Hamiltonian assumes a relativistic form.

What this theorem says in effect is that for any system satisfying the conditions required for a multi-channel scattering system, there is only one set of free Hamiltonians which will describe the asymptotic behaviour of the fragments.

It is possible now to deduce some properties of the wave operators, most of which are easily proved:

(i) $\Omega_\pm^{(\alpha)}$ are partial isometries.

[A linear operator Ω is a partial isometry if $E \equiv \Omega^\dagger \Omega$ is a projection, i.e. $E^2 = E$ and $E = E^\dagger$. Then so is $F \equiv \Omega \Omega^\dagger$ a projection. If E and F are projections on the sets M and N respectively, then $EM = N$ and $\|Ef\| = \|f\|$ for f in M and $Ef = 0$ for f in $M^\perp$ (hence the reason for the terminology)]

i.e. $$\Omega_\pm^{(\alpha)\dagger} \Omega_\pm^{(\alpha)} = E^{(\alpha)} \qquad (74)$$

$$\Omega_\pm^{(\alpha)} \Omega_\pm^{(\alpha)\dagger} = F_\pm^{(\alpha)} \qquad (75)$$

(For $E^{(\alpha)}$ the plus and minus cases are identical).

(ii) The projections $F_{\pm}^{(\alpha)}$ are orthogonal for different channels.

$$\text{i.e.} \quad F_{+}^{(\alpha)} F_{+}^{(\beta)} = F_{-}^{(\alpha)} F_{-}^{(\beta)} = 0 \qquad \text{if } \alpha \neq \beta \tag{76}$$

$$\text{(iii)} \quad E_N = \sum_{\alpha} F_{+}^{(\alpha)} = \sum_{\beta} F_{-}^{(\beta)} \tag{77}$$

where E_N is the projection operator with range N .

$$\text{(iv)} \quad \Omega_{+}^{(\alpha)\dagger} \Omega_{+}^{(\beta)} = \Omega_{-}^{(\alpha)\dagger} \Omega_{-}^{(\beta)} = E^{(\alpha)} \delta_{\alpha\beta} \tag{78}$$

$$\text{(v)} \quad V_t \Omega_{\pm}^{(\alpha)} = \Omega_{\pm}^{(\alpha)} U_t^{(\alpha)} \tag{79}$$

$$\text{(vi)} \quad \left[F_{\pm}^{(\alpha)}, V_t \right] = 0 \qquad \text{for all } t \text{ and all } \alpha \tag{80}$$

2.8 The multi-channel scattering operator.

We must now discuss the construction of the scattering operator. As we saw in the first chapter there is no generalisation of the operator $S = \Omega_{-}^{\dagger} \Omega_{+}$ for the multi-channel case. However it is possible to define a different scattering operator by the relation

$$S' = \Omega_{+} \Omega_{-}^{\dagger} \tag{81}$$

which will permit such a generalisation.

It is then easy to show that

$$S' S'^{+} = E_N \tag{82}$$

$$S'^{+} S' = E_N \tag{83}$$

$$E_N S' = S' \tag{84}$$

$$S' E_N = S' \tag{85}$$

so that S' is unitary in the invariant subspace N .

Directly from the definitions we obtain the following relationships between S and S' :-

$$\Omega_+^{+} S' \Omega_+ = \Omega_-^{+} S' \Omega_- = S \tag{86}$$

$$\Omega_+^{+} S' \Omega_- = I \tag{87}$$

$$\Omega_-^{+} S' \Omega_+ = S^2 \tag{88}$$

Now S' commutes with V_t (by using the intertwining property) but not with U_t and so

$$S'(t) \equiv U_t^{+} S' U_t \tag{89}$$

does have an explicit dependence on t . It may be shown to have the strong limit

$$\lim_{t \to \pm\infty} S'(t) = S \tag{90}$$

If we go over to the multi-channel case we can define a sequence of operators

$$S'_n = \sum_{\alpha=1}^{n} \Omega_+^{(\alpha)} \Omega_-^{(\alpha)\dagger} \tag{91}$$

and investigate the limit as $n \to \infty$ (of course if the number of channels is finite this difficulty does not arise). It may be shown that this limit exists in the strong sense and defines the scattering operator

$$S' = \sum_{\alpha=1}^{\infty} \Omega_+^{(\alpha)} \Omega_-^{(\alpha)\dagger} \tag{92}$$

It has the following properties which may be proved very simply:-

(i) It is quasi-unitary

i.e. $$S'^{\dagger} S' = S' S'^{\dagger} = E_N \tag{93}$$

(ii) $$[V_t, S'] = 0 \tag{94}$$

(iii) $$\Omega_-^{(\beta)\dagger} S' \Omega_-^{(\alpha)} = \Omega_+^{(\beta)\dagger} S' \Omega_+^{(\alpha)} = \Omega_-^{(\beta)\dagger} \Omega_+^{(\alpha)} \tag{95}$$

The physical interpretation of the scattering operator as thus defined may be obtained as follows:-

Let the system in the distant past be in the channel α

which means that its state vector is given by $U_t^{(\alpha)} f_\alpha$ where f_α is in D_α. Because of the basic property of scattering systems this state vector has the same limit in the norm as $V_t f_+^{(\alpha)}$ as $t \to -\infty$. But this will also describe its development through all time and so the probability of finding the system in the distant future in the state $U_t^{(\beta)} f_\beta$ (with f_β in D_β) is given by

$$P_{\beta\alpha} = \lim_{t\to+\infty} \left|(U_t^{(\beta)} f_\beta,\ V_t f_+^{(\alpha)})\right|^2$$

$$= \lim_{t\to+\infty} \left|(V_t^\dagger U_t^{(\beta)} f_\beta,\ f_+^{(\alpha)})\right|^2$$

$$= \left|(f_-^{(\beta)},\ f_+^{(\alpha)})\right|^2 \tag{96}$$

and this expression may be rewritten in equivalent forms by means of the relationships

$$(f_-^{(\beta)}, f_+^{(\alpha)}) = (f_-^{(\beta)}, S' f_-^{(\alpha)}) \tag{97}$$

$$= (f_+^{(\beta)}, S' f_+^{(\alpha)}) \tag{98}$$

$$= (f_\beta, \Omega_-^{(\beta)\dagger} S' \Omega_-^{(\alpha)} f_\alpha) \tag{99}$$

$$= (f_\beta, \Omega_+^{(\beta)\dagger} S' \Omega_+^{(\alpha)} f_\alpha) \tag{100}$$

We hence see the physical relationship of the new scattering operator to the transition probability; from the above expressions we are led to an expression for the cross-section in the usual way.

2.9 Integral representations and equations.

Tixaire[24] has shown how, in the multi-channel case, to construct integral representations for the wave operators and hence integral equations for the scattering states. Here we merely reproduce his results.

Integral representations for the wave operators $\Omega_{\pm}^{(\alpha)}$, $\Omega_{\pm}^{(\alpha)+}$

(i) Riemann integral representations.

Suppose $g_+(t,\epsilon) \geq 0$ for $-\infty < t \leq 0$ and $0 < \epsilon \leq \epsilon_0$, that it is continuous in t and that $\int_{-\infty}^{0} g_+(t,\epsilon)\,dt = 1$ for all permissible ϵ (and we have similar assumptions for $g_-(t,\epsilon)$)

Then the following integral representations exist:-

$$\Omega_{\pm}^{(\alpha)} = \lim_{\epsilon \to 0} \mp \int_{0}^{\mp\infty} g_{\pm}(t,\epsilon)\, V_t^{+} U_t^{(\alpha)}\, dt \qquad (101)$$

with domain $D_{\pm}^{(\alpha)}$

$$\Omega_{\pm}^{(\alpha)\dagger} = \lim_{\epsilon \to 0} \pm \int_{\mp\infty}^{0} g_{\pm}(t,\epsilon)\, U_t^{(\alpha)\dagger} V_t \, dt \tag{102}$$

with domain $R_{\pm}^{(\alpha)}$

(We note that these correspond with the result obtained by Jauch for the single channel case when we take

$g_{\pm}(t,\epsilon) = \epsilon\, e^{\pm \epsilon t}$)

(ii) Cauchy integral representation.

If σ_α is any compact subset of the spectral set* of H_α and $M(\sigma_\alpha)$ is the corresponding closed linear manifold, then we will have

$$U_t^{(\alpha)} = \frac{1}{2\pi i} \int_{C(\sigma_\alpha)} e^{-izt} R(z, H_\alpha)\, dz \tag{103}$$

with domain $M(\sigma_\alpha)$

by Cauchy's theorem, where $R(z, H_\alpha) \equiv (z - H_\alpha)^{-1}$ is the resolvent of H_α at z, and $C(\sigma_\alpha)$ is a closed contour in the complex z-plane which contains σ_α. If we now assume the existence of some contour C such that for a fixed $g_+(t,\epsilon)$ and any compact subset σ_α we are assured of the existence of

$$\int_{-\infty}^{0} g_+(t,\epsilon)\, e^{-izt} V_t^{\dagger} \, dt \equiv N_+(g_+, z) \tag{104}$$

and

---

* For a discussion of the spectral set and the resolvent see, for example, reference (16), page 128.

$$\int_{-\infty}^{0} g_+(t,\epsilon) e^{-izt} U_t^{(\alpha)\dagger} dt \equiv N_+^{(\alpha)}(g_+, z) \tag{105}$$

for every z enclosed by C and ϵ in $0 < \epsilon \leq \epsilon_0$. Then N_+ and $N_+^{(\alpha)}$ are strongly continuous on C considered as functions of z and substitution in the Riemann integral representation given above yields

$$\Omega_{\pm}^{(\alpha)} = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_C N_{\pm}(g_{\pm}, z) R(z, H_\alpha) dz \tag{106}$$

with domain the intersection of $M(\sigma_\alpha)$ and $D_{\pm}^{(\alpha)}$

and $$\Omega_{\pm}^{(\alpha)\dagger} = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_C N_{\pm}^{(\alpha)}(g_{\pm}, z) R(z, H) dz \tag{107}$$

with domain $M(\sigma) \cap R_{\pm}^{(\alpha)}$

In the special case $g_{\pm}(t,\epsilon) = \epsilon e^{\pm \epsilon t}$ we get

$$\Omega_{\pm}^{(\alpha)} = \lim_{\epsilon \to 0} \pm \frac{\epsilon}{2\pi} \int_C R(z \pm i\epsilon, H) R(z, H_\alpha) dz \tag{108}$$

with domain $M(\sigma_\alpha) \cap D_{\pm}^{(\alpha)}$

and $$\Omega_{\pm}^{(\alpha)\dagger} = \lim_{\epsilon \to 0} \pm \frac{\epsilon}{2\pi} \int_C R(z \pm i\epsilon, H_\alpha) R(z, H) dz \tag{109}$$

with domain $M(\sigma) \cap R_{\pm}^{(\alpha)}$

where we may choose C as being inside the strip $|\mathrm{Im}\, z| \leq \frac{\epsilon}{2}$

(iii) Stieltjes integral representations.

Here we use the spectral resolution of the total and channel Hamiltonians:

$$H_\alpha = \int \lambda \, dE_\alpha(\lambda) \qquad H = \int \lambda \, dE(\lambda) \tag{110}$$

understood, as usual, in the Stieltjes integral sense. Then Tixaire shows that we have the integral representations

$$\Omega_\pm^{(\alpha)} = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} N_\pm(g_\pm, \lambda)\, dE_\alpha(\lambda) \tag{111}$$

with domain $D_\pm^{(\alpha)}$

$$\Omega_\pm^{(\alpha)\dagger} = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} N_\pm^{(\alpha)}(g_\pm, \lambda)\, dE(\lambda) \tag{112}$$

with domain $R_\pm^{(\alpha)}$

or, taking the particular form $g_\pm(t, \epsilon) = \epsilon e^{\pm \epsilon t}$

$$\Omega_\pm^{(\alpha)} = \lim_{\epsilon \to 0} \pm i\epsilon \int_{-\infty}^{\infty} R(\lambda \pm i\epsilon, H)\, dE_\alpha(\lambda) \tag{113}$$

with domain $D_\pm^{(\alpha)}$

$$\Omega_\pm^{(\alpha)\dagger} = \lim_{\epsilon \to 0} \pm i\epsilon \int_{-\infty}^{\infty} R(\lambda \pm i\epsilon, H_\alpha)\, dE(\lambda) \tag{114}$$

with domain $R_\pm^{(\alpha)}$

Tixaire now applies the above results to obtain scattering integral equations which contain the interaction Hamiltonian $H_I^{(\alpha)} \equiv H - H_\alpha$ explicitly. In most physical cases this is treated as a small perturbation, and approximations made accordingly. However, it may be unbounded (e.g. Coulomb or Yukawa potential) and this obviously would lead to difficulties. These we circumvent by the following conditions (which admit a wide range of interaction Hamiltonians).

We suppose that there exists, for each $H_I^{(\alpha)}$ a domain $\tilde{D}_+^{(\alpha)}$ everywhere dense in $D_+^{(\alpha)}$ such that

(a) $\tilde{D}_+^{(\alpha)} \subseteq D_{H_0}$, $U_t^{(\alpha)}\tilde{D}_+^{(\alpha)} \subseteq D_H, D_{H_I^{(\alpha)}}$ for $-\infty < t \leq 0$

(b) $H_I^{(\alpha)} U_t^{(\alpha)}$ is strongly continuous on $\tilde{D}_+^{(\alpha)}$ as a function of t for $-\infty < t \leq 0$

We also assume that, for some $g_+(t, \epsilon)$

(c) $\int_{-\infty}^{0} g_+^1(t,\epsilon) \| H_I^{(\alpha)} U_t^{(\alpha)} f \| dt < \infty$ for every f in $\tilde{D}_+^{(\alpha)}$

where $g_+^1(t,\epsilon) \equiv \int_{-\infty}^{t} g_+(t,\epsilon)dt$

We have similar considerations for $g_-(t,\epsilon)$ and the adjoint operators (which lead to the existence of ranges $\tilde{R}_\pm^{(\alpha)}$ under conditions similar to the above for domains).

(i) Riemann integral equations.

Applying the above ideas to the Riemann integral representations of the wave operators (101) and (102) we obtain immediately

$$\Omega_\pm^{(\alpha)} = 1 - i \lim_{\epsilon \to 0} \int_{\mp\infty}^{0} g_\pm^1(t,\epsilon) V_t^\dagger H_I^{(\alpha)} U_t^{(\alpha)} dt \qquad (115)$$

with domain $\tilde{D}_\pm^{(\alpha)}$

$$\Omega_{\pm}^{(\alpha)+} = 1 + i \lim_{\epsilon\to 0} \int_{\mp\infty}^{0} g'_{\pm}(t,\epsilon)\, U_t^{(\alpha)+} H_I^{(\alpha)} V_t \, dt \tag{116}$$

with domain $\tilde{R}_{\pm}^{(\alpha)}$

If we now apply equation (116) to an element $f_{\pm}^{(\alpha)}$ in $\tilde{R}_{\pm}^{(\alpha)}$ we get (noting that $\Omega_{\pm}^{(\alpha)+} f_{\pm}^{(\alpha)} = f_\alpha$)

$$f_{\pm}^{(\alpha)} = f_\alpha - i \lim_{\epsilon\to 0} \int_{\mp\infty}^{0} g'_{\pm}(t,\epsilon)\, U_t^{(\alpha)+} H_I^{(\alpha)} V_t f_{\pm}^{(\alpha)} \, dt \tag{117}$$

which are the required scattering integral equations.

Taking again the special case $g_{\pm}(t,\epsilon) = \epsilon e^{\pm\epsilon t}$ yields

$$f_{\pm}^{(\alpha)} = f_\alpha - i \lim_{\epsilon\to 0} \int_{\mp\infty}^{0} e^{\pm\epsilon t}\, U_t^{(\alpha)+} H_I^{(\alpha)} V_t f_{\pm}^{(\alpha)} \, dt \tag{118}$$

(ii) Cauchy integral equations.

Here we assume that

$$\int_{-\infty}^{0} g'_{+}(t,\epsilon)\, e^{-izt} V_t^{+} \, dt \equiv N'_{+}(g_{+}, z)$$

and $H_I^{(\alpha)} R(z, H_\alpha)$ exist, and are strongly continuous functions of z within C on a subset $\tilde{D}_{\pm}^{(\alpha)}(\sigma_\alpha)$ everywhere dense in $\tilde{D}_{\pm}^{(\alpha)} \cap M(\sigma_\alpha)$ for every ϵ such that $0 < \epsilon \leq \epsilon_0$

We then obtain the Cauchy integral equations from the Cauchy integral representation in a similar manner to the Riemann case above. For the special case $g_{\pm}(t,\epsilon) = \epsilon e^{\pm\epsilon t}$ they reduce to

$$f_{\pm}^{(\alpha)} = f_\alpha + \frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_{C} R(z \pm i\epsilon, H_\alpha)\, H_I^{(\alpha)} R(z, H)\, dz \, f_{\pm}^{(\alpha)} \tag{119}$$

where C may be taken in the region $|\mathfrak{Im}\, z| \leq \frac{\epsilon}{2}$.

(iii) Stieltjes integral equations.

Under suitable restrictions and with the special weight factor $g_\pm(t,\epsilon) = \epsilon e^{\pm\epsilon t}$ we obtain the equations

$$f_\pm^{(\alpha)} = f_\alpha + \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{\lambda + i\epsilon - H_\alpha} H_I^{(\alpha)} dE(\lambda) f_\pm^{(\alpha)} \qquad (120)$$

In spite of the rather complicated nature of the restrictions imposed to justify the above expressions, it is felt likely that they will be satisfied in many actual physical situations; in particular for scattering by potentials $V(\underline{x})$ belonging to $L^2(E_3)$ all our conditions are satisfied.

This concludes our survey of the mathematically meaningful approach to scattering theory as introduced by Jauch. It is at once apparent how much more satisfying this approach is.

CHAPTER III

DISPERSION RELATIONS AND THE MANDELSTAM REPRESENTATION FOR POTENTIAL SCATTERING.

With the present field-theoretic interest in dispersion relations and the Mandelstam representation, it is thought to be pertinent to give a brief account of these topics as they appear in the theory of potential scattering in non-relativistic quantum mechanics. The benefits of this are that this problem, concerning which so much is known, might throw some light on the field theoretic case, where we are relatively in the dark.

3.1 Dispersion relations.

We shall first of all discuss the derivation of the dispersion relations in the case of Schrödinger scattering, as derived by Khuri.[25]

The Schrödinger equation written in its dimensionless form is

$$\left(\nabla^2 + k^2 - \lambda V(r)\right)\psi(\underline{x}) = 0 \qquad (1)$$

and we look for a solution, which at large distances behaves like a plane wave, together with an outgoing spherical wave, i.e.

$$\psi(\underline{x}) = e^{i\underline{k}.\underline{x}} + \lambda\int K(\underline{x},\underline{y})\,\psi(\underline{y})\,d^3\underline{y} \qquad (2)$$

where
$$K(\underline{x},\underline{y}) = -V(y)\,\frac{e^{ik|\underline{x}-\underline{y}|}}{4\pi|\underline{x}-\underline{y}|} \tag{3}$$

is the appropriate Green's function to give the correct asymptotic form. (We use the notation $y = |\underline{y}|$ etc.)

Asymptotically we have

$$\psi(\underline{x}) = e^{i\underline{k}.\underline{x}} + \frac{1}{x}e^{ikx}f(k,\tau) \tag{4}$$

where $f(k,\tau)$ is the scattering amplitude and $\tau = k[2(1-\cos\theta)]^{1/2}$ is the magnitude of the momentum transfer (we note that $\underline{k}' = k\,\underline{x}/x$ and $\cos\theta = \frac{1}{k^2}\underline{k}.\underline{k}'$).

We now apply the Fredholm theory of integral equations (first used in this connection by Jost and Pais[26]) to equation (2). The conditions on the potential

$$\int_0^\infty r^2|V(r)|\,dr < \infty \qquad \int_0^\infty r\,|V(r)|\,dr < \infty \tag{5}$$

will ensure absolute uniform convergence of the series occurring in the Fredholm solution.

Defining the new variables

$$\begin{aligned} \underline{\pi} &= \tfrac{1}{2}(\underline{k}+\underline{k}') & \underline{r} &= \underline{y}-\underline{z} \\ \underline{\tau} &= \underline{k}-\underline{k}' & \underline{R} &= \tfrac{1}{2}(\underline{y}+\underline{z}) \\ \underline{n} &= \underline{\pi}/\pi \end{aligned} \tag{6}$$

the Fredholm theory eventually leads us to the result that

$$f(k,\tau) = -\frac{\lambda}{4\pi}\tilde{V}(\tau) + G_2(k,\tau) + G_3(k,\tau) + \frac{G_4(k,\tau)}{\Delta(\lambda^2,k)} + \frac{G_5(k,\tau)}{\Delta(\lambda^2,k)} \tag{7}$$

where $\Delta(\lambda^2, k)$ is the usual Fredholm denominator:

$$\Delta(\lambda^2,k) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda^2)^n}{n!} \int d^3\underline{x}_1 \cdots \int d^3\underline{x}_n \, D^{(n)}(k;\underline{x}_1 \cdots \underline{x}_n) \tag{8}$$

$$D^{(n)}(k;\underline{x}_1 \cdots \underline{x}_n) = \begin{vmatrix} K_2(\underline{x}_1,\underline{x}_1) & \cdots & K_2(\underline{x}_1,\underline{x}_n) \\ \vdots & & \vdots \\ K_2(\underline{x}_n,\underline{x}_1) & \cdots & K_2(\underline{x}_n,\underline{x}_n) \end{vmatrix} \tag{9}$$

$$K_2(\underline{x},\underline{y}) = \int K(\underline{x},\underline{z})\, K(\underline{z},\underline{y})\, d^3\underline{z} \tag{10}$$

and

$$G_j(k,\tau) = -\frac{\lambda^j}{4\pi}\int e^{i(k^2-\frac{1}{4}\tau^2)^{\frac{1}{2}}\underline{n}\cdot\underline{r}} N_j(\underline{R}-\tfrac{1}{2}\underline{r}, \underline{R}+\tfrac{1}{2}\underline{r}) \times \tag{11}$$

$$\times\, e^{i\underline{\tau}\cdot\underline{R}}\, d^3\underline{r}\, d^3\underline{R} \qquad j = 2-5$$

where

$$N_2(\underline{z},\underline{y}) = V(z)\, K(\underline{z},\underline{y}) \tag{12}$$

$$N_3(\underline{z},\underline{y}) = V(z)\, K_2(\underline{z},\underline{y}) \tag{13}$$

$$N_4(\underline{z},\underline{y}) = V(z)\int K(\underline{z},\underline{x}_1)\,\Delta(\lambda^2,k;\underline{x}_1,\underline{y})\, d^3\underline{x}_1 \tag{14}$$

$$N_5(\underline{z},\underline{y}) = V(z)\int K(\underline{z},\underline{x}_1)\,\Delta(\lambda^2,k;\underline{x}_1,\underline{x}_2)\, K(\underline{x}_2,\underline{y})\, d^3\underline{x}_1\, d^3\underline{x}_2 \tag{15}$$

and $$\Delta(\lambda^2, k; \underline{x}, \underline{y}) = K_2(\underline{x}, \underline{y}) + \sum_{n=1}^{\infty} \frac{(-\lambda^2)^n}{n!} \int d^3\underline{x}_1 \cdots \int d^3\underline{x}_n \times \tag{16}$$

$$\times B^{(n)}(k; \underline{x}, \underline{y}, \underline{x}_1, \cdots \underline{x}_n)$$

with

$$B^{(n)}(k; \underline{x}, \underline{y}, \underline{x}_1 \cdots \underline{x}_n) = \begin{vmatrix} K_2(\underline{x},\underline{y}) & K_2(\underline{x},\underline{x}_1) & \cdots & K_2(\underline{x},\underline{x}_n) \\ K_2(\underline{x}_1,\underline{y}) & K_2(\underline{x}_1,\underline{x}_1) & \cdots & K_2(\underline{x}_1,\underline{x}_n) \\ \vdots & \vdots & & \vdots \\ K_2(\underline{x}_n,\underline{y}) & K_2(\underline{x}_n,\underline{x}_1) & \cdots & K_2(\underline{x}_n,\underline{x}_n) \end{vmatrix} \tag{17}$$

Also $$\tilde{V}(\tau) = \int e^{i\underline{\tau}\cdot\underline{y}} V(y)\, d^3\underline{y} \tag{18}$$

It is clear that $N_j(k) = N_j^*(-k)$ and we hence observe that this implies

$$G_j(k, \tau) = G_j^*(-k, \tau) \tag{19}$$

Now the above integrals and series, under the conditions (5) on the potential all converge for physical scattering (by physical scattering we mean that we have a real scattering angle θ, and hence, from the expression for momentum transfer $\tau = k[2(1-\cos\theta)]^{1/2}$, we see that the condition for physical scattering is that we should have $|k| \geq \frac{1}{2}\tau$; it is under this condition that we obtain convergence). However, in the dispersion relation approach we allow k to be complex and vary in the complex plane; the integrals and series will not converge for $|k| \leq \frac{1}{2}\tau$ ($\tau \neq 0$) unless the potential falls off fast enough.

Now, for dispersion relations what we want are the analytic properties of the scattering amplitude and, in fact, Khuri proves the following result:-

If the potential $V(r)$ satisfies

(a) $|V(r)| \leq M'/r^2$

(b) $\int_0^\infty r|V(r)|\,dr \leq M < \infty$

(c) $\int_0^\infty e^{\alpha r} r^2 |V(r)|\,dr \leq L < \infty$ (20)

then for real $\tau \leq 2\alpha$ the $G_j(k,\tau)$ are analytic functions of k regular for $\mathrm{Im}\,k > 0$ and uniformly bounded for $\mathrm{Im}\,k \geq 0$. On the real axis there are branch points at $k = \pm\frac{1}{2}\tau$.

For the proof of this we need the following theorem* :-

If $\Phi(k,x)$ is regular in a region Γ in the k plane and continuous in the closed region of Γ and its boundary B then if $f(k) = \int_\Lambda \Phi(k,x)\,dx$, $f(k)$ is analytic in k, regular and uniformly bounded in Γ provided that there exists $\Psi(x)$ such that $|\Phi(k,x)| \leq \Psi(x)$ for all k on B and $\int_\Lambda \Psi(x)\,dx < \infty$.

In the case under consideration we take the curve B as

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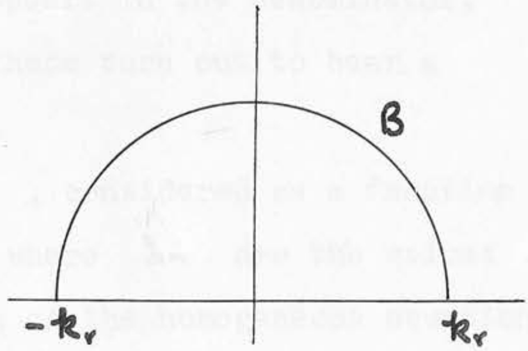
* See reference (27) pp. 99-100.



shown, show that the integrands in the defining relations for the $G_j(k,\tau)$ are analytic in k within B and that for k on B they are bounded by a function of $\underline{r}$, $\underline{R}$ and τ which is integrable if $\frac{1}{2}\tau \leq \alpha$. The proofs of these facts are somewhat lengthy, and will be omitted; we shall be content with noticing that

(i) the conditions (20) are used in the proof

(ii) the case of $G_2(k,\tau)$ is comparatively simple and does not involve the use of the above theorem.

(iii) for $j = 3, 4, 5$ we need to prove $K_2(\underline{x},\underline{y})$ and $\Delta(\lambda^2, k; \underline{x}, \underline{y})$ bounded in $\operatorname{Im} k \geq 0$

(iv) the factor $e^{i(k^2-\frac{1}{4}\tau^2)^{1/2}\underline{n}\cdot\underline{r}}$ leads to the branch points at $k = \pm\frac{1}{2}\tau$.

(v) it is also shown that for $\operatorname{Im} k \geq 0$

$$\lim_{|k|\to\infty} \Delta(\lambda^2, k; \underline{x}, \underline{y}) = 0$$

To obtain complete information on the analytic properties of the scattering amplitude it is still necessary to know those of the function $\Delta(\lambda^2, k)$. It is not difficult to show that this is an analytic function of k regular for $\operatorname{Im} k > 0$ and uniformly bounded for $\operatorname{Im} k \geq 0$. However we are also interested, since $\Delta(\lambda^2, k)$ appears in the denominator, in the zeros of this function. These turn out to bear a relationship to the bound states.

Now, the zeros of $\Delta(\lambda^2, k)$, considered as a function of λ, occur for $\lambda = \pm\lambda_n$ where λ_n are the values for which there exists a solution of the homogeneous equation

$$\psi_n(\underline{k}, \underline{x}) = \lambda_n \int K(\underline{x}, \underline{y}) \, \psi_n(\underline{k}, \underline{y}) \, d^3\underline{y} \tag{21}$$

It is then possible to take out a factor from

$$\frac{\Delta(\lambda^2, k; \underline{x}, \underline{y})}{\Delta(\lambda^2, k)} \quad \text{to give} \quad \frac{\Delta_1(\lambda, k; \underline{x}, \underline{y})}{\Delta_1(\lambda, k)} \quad \text{such}$$

that the numerator and denominator have no zeros in common. This will not affect convergence and analytic properties:

$$\frac{G_4}{\Delta(\lambda^2, k)} \quad \text{becomes} \quad \frac{G_4^{(1)}}{\Delta_1(\lambda, k)} \quad \text{with the same}$$

analytic properties, and the same result holds also for

$$\frac{G_5}{\Delta(\lambda^2, k)}$$

So, if $\Delta_1(\lambda, k) = 0$ there exists at least one solution of the homogeneous equation

$$\psi_n(\underline{k}_n, \underline{x}) = \lambda \int K(\underline{x}, \underline{y}) \, \psi_n(\underline{k}_n, \underline{y}) \, d^3\underline{y} \tag{22}$$

Now, for real $\underline{k}_n \neq 0$ Jost and Pais showed that λ in this equation must be complex. Hence, for real λ, $\Delta_1(\lambda, k)$

can have no zeros on the real axis of k (except possibly at $k = 0$). For $\mathrm{Im}\, k > 0$ it can be shown that all zeros are on the positive imaginary axis, and that $\psi_n(\underline{k}_n, \underline{x})$ are the bound states (the possible zero at the origin may or may not correspond to a bound state).

Further, the fact that $\int_0^\infty r|V(r)|\,dr \leq M < \infty$

is true ensures that the number of bound states is finite and the k_n will have a finite maximum, corresponding to the lowest energy state.

The asymptotic behaviour of $\Delta(\lambda^2, k)$ may be found quite easily:

$$\lim_{|k| \to \infty} \Delta(\lambda^2, k) = 1 \tag{23}$$

for $\mathrm{Im}\, k \geq 0$

The asymptotic properties of $G_j(k, \tau)$ (j = 3, 4, 5) are determined by those of $\Delta(\lambda^2, k; \underline{x}, \underline{y})$ and $\Delta(\lambda^2; k)$;

we have in fact
$$\lim_{|k| \to \infty} G_j(k, \tau) = 0 \tag{24}$$

for $\mathrm{Im}\, k \geq 0$

Also the Riemann-Lebesgue lemma will show that $G_2(k, \tau)$ will vanish in the limit of large $\mathrm{Re}\, k$, which is enough to ensure that the Cauchy integral of G_2 round a semi-circle in the upper half plane may be made arbitrarily small by making the semi-circle large enough.

We now have enough information to obtain the required dispersion relation.

Writing

$$g(k,\tau) = G_2(k,\tau) + G_3(k,\tau) + \frac{G_4^{(1)}(k,\tau)}{\Delta_1(\lambda,k)} + \frac{G_5^{(1)}(k,\tau)}{\Delta_1(\lambda,k)} \qquad (25)$$

we have

$$g(k,\tau) = f(k,\tau) + \frac{\lambda}{4\pi}\tilde{V}(\tau) \qquad (26)$$

and the results we have noted so far are as follows:-

(i) $g(k,\tau)$, for $\tau \leq 2\alpha$, is analytic in k, regular in $\mathrm{Im}\, k > 0$, continuous and uniformly bounded for $\mathrm{Im}\, k \geqslant 0$ apart from

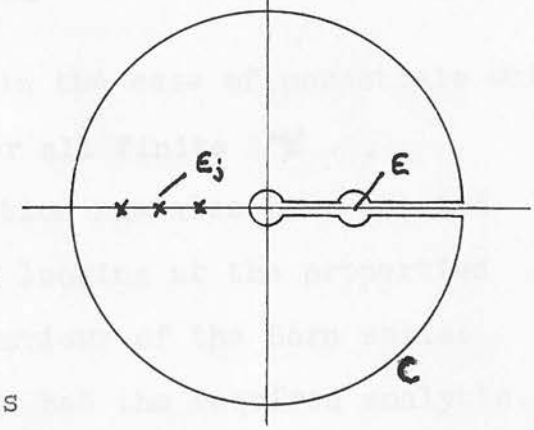
(ii) a finite number of poles at the zeros of $\Delta_1(\lambda,k)$ which for real λ all lie on the positive imaginary axis.

(iii) $g(k,\tau)$ has branch points on the real axis at $k = \pm\frac{1}{2}\tau$

(iv) $g(k,\tau)$ behaves as $|k| \to \infty$ in such a way that its Cauchy integral vanishes round an infinite semi-circle in the upper half k. plane.

To prove the dispersion relation we utilise the energy variable E, where $E = k^2$ with our present system of units. Writing $g(E,\tau)$ for the expression (25) (where, of course $g(E,\tau)$ is of a different functional form than $g(k,\tau)$)

we have that in the complex E plane $g(E,\tau)$ is analytic everywhere except for a cut on the positive real axis and a finite number of poles on the negative real axis. Applying Cauchy's theorem to the integral of $g(E,\tau)$ round the contour C shown we obtain



$$\int_C \frac{g(E',\tau)}{E'-E}\,dE' = 2\pi i \sum_j \frac{R_j(\tau)}{E_j - E} \qquad (27)$$

where the $R_j(\tau)$ are the residues of $g(E,\tau)$ at the bound states E_j ; Taking the limit as the large circle expands to infinity, the small circles contract to zero and the two lines approach the real axis, we obtain,

using the fact that $$g(E+i\epsilon,\tau) = g^*(E-i\epsilon,\tau) \qquad (28)$$
$$(E \geq 0)$$

the result that

$$\mathrm{Re}\, g(E,\tau) = \frac{1}{\pi} P\int_0^\infty \frac{\mathrm{Im}\, g(E',\tau)}{E'-E}\,dE' + \sum_{j=0}^{N} \frac{R_j(\tau)}{E-E_j} \qquad (29)$$

(with $E_0 = 0$)

where P , as usual, means that we take the principal value of the integral. Then, returning to the scattering amplitude considered as a function of the energy, $f(E,\tau)$, we obtain the required dispersion relation:

$$\mathrm{Re}\, f(E,\tau) = \frac{1}{\pi} P \int_0^\infty \frac{\mathrm{Im}\, f(E',\tau)}{E'-E} dE' + \sum_{j=0}^{N} \frac{R_j(\tau)}{E-E_j} - \frac{\lambda}{4\pi} \tilde{V}(\tau) \qquad (30)$$

This holds for $\frac{1}{2}\tau \leq \alpha$, and so, in the case of potentials which fall off like Gaussian or faster, for all finite τ.

The proof of a dispersion relation has also been carried out by Klein and Zemach(28) who, by looking at the properties of the Green's function and the behaviour of the Born series showed that the scattering amplitude had the required analytic properties. An investigation was also performed by Gasiorowicz and Noyes(29) who obtained the result using an approach akin to a field-theoretic method.

Now, in the above dispersion relation (30), we require an integral over the unphysical region $0 < E' < \frac{1}{4}\tau^2$ where we have no experimental data on $f(E,\tau)$ to put into our equation. However, Khuri(25) showed that the partial wave expansion of $f(E,\tau)$ is convergent in the unphysical region if $\tau < \alpha$ and can then be used to define $f(E,\tau)$ in this region and so give all the information required in the dispersion relation above.

Klein and Zemach(28) succeeded in proving a stronger result than this: that, in fact, the partial wave expansion was convergent in the unphysical region for $\tau < 2\alpha$, namely the same range of validity as the dispersion relation itself.

It is interesting to note that in the forward direction $(\tau = 0)$ the above relation will hold even if $\alpha = 0$. This means that condition (20)(c) is much weakened and leaves us with a large class of potentials.

An analogous calculation was carried out by Khuri and Treiman[30] for the case of Dirac potential scattering, i.e. the scattering of a particle obeying the Dirac equation by a central potential. Here the discussion proceeds by way of the T-matrix (which for the Schrödinger case is essentially the same as the scattering amplitude $f(k, \tau)$) which will be an operator in spinor space. It is shown that this T-matrix, extended to the complex energy plane is analytic in the plane cut along the real axis from m (the mass of the particle concerned) to $+\infty$ and from $-m$ to $-\infty$, apart from poles (also on the real axis) corresponding to bound states. These properties, together with certain asymptotic and symmetry properties lead to the required dispersion relation on performing a Cauchy integration round a suitable contour.

3.2 Mandelstam Representation.

The problem of the Mandelstam representation in potential scattering has been treated by Blankenbecler et al.[31] What we have indicated above is that a dispersion relation in the energy variable holds for values of the squared momentum transfer $t = \tau^2$ less than $4\alpha^2$. For a Mandelstam representation we are also interested in the analyticity of the scattering amplitude in the t-plane.

We consider potentials subject to the restrictions

$$\text{(a)} \quad |V(r)| < \frac{M}{r^2}$$

$$\text{(b)} \quad \int_{M'}^{\infty} r^2 |V(r)| \, dr < \infty$$

$$\text{(c)} \quad \int_{0}^{M''} r |V(r)| \, dr < \infty \tag{31}$$

where M, M', M'' are finite positive numbers.

We also assume that the potential has the representation

$$r V(r) = \int_0^\infty d\mu \, \sigma(\mu) \, e^{-\mu r} \tag{32}$$

which, roughly speaking, means that we have a superposition of Yukawa-type potentials.

If, further,

$$\int_0^\infty r |V(r)| e^{\alpha r} \, dr < \infty \qquad \text{for} \quad 0 \le \alpha \le m \tag{33}$$

then $V(r)$ is said to have a range $1/m$ and it is easily seen that $\sigma(\mu) = 0$ for $\mu < m$. The conditions (31) (b) and (c) above then imply that

(i) $$\lim_{\mu \to 0} \frac{\sigma(\mu)}{\mu} = 0$$

(ii) $$\lim_{\mu \to \infty} \sigma(\mu) = 0 \qquad (34)$$

It is found to be a convenient procedure to consider the first Born approximation to the scattering amplitude separately. This term is given by*

$$f_B(t) = -\frac{M}{\hbar^2 . 2\pi} \int d^3\underline{r}\; e^{i \underline{\tau} . \underline{r}}\, V(r)$$

$$= -\frac{2M}{\hbar^2} \int_0^\infty d\mu \frac{\sigma(\mu)}{\mu^2 + t} \qquad (35)$$

where M is the reduced mass of the system. $f_B(t)$ is obviously analytic in the t-plane cut along the real axis from $-\infty$ to $-m^2$ (since essentially the lower limit of the above integral is $m \geqslant 0$). If, however, $\sigma(\mu)$ contained a δ -function then we should have an isolated pole rather than a branch cut. Because of the condition (34) (ii) $f_B(t) \to 0$ as $t \to \infty$

We choose units such that $\hbar^2/2M = 1$ and then the energy is just given by $E = k^2$; we consider the scattering amplitude as a function of E and t : $f(E, t)$.

Now we know that the scattering amplitude (related to

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* Reference (10) page 163.

elements of the T -matrix by $T_{\underline{k}'\underline{k}} = \frac{k}{2\pi i} f(\underline{k}',\underline{k})$) will satisfy the equation

$$f(\underline{k}',\underline{k}) = f_B(\underline{k}'-\underline{k}) - \frac{1}{(2\pi)^3}\int d^3k_1 \, \frac{\tilde{V}(\underline{k}'-\underline{k}_1)\, f(\underline{k}_1,\underline{k})}{k_1^2 - k^2 - i\epsilon} \qquad (36)$$

(where $\tilde{V}(\underline{q})$ is the Fourier transform of $V(r)$) which is readily derived from the Lippmann-Schwinger equation.

A solution of equation (36) is given by

$$f(\underline{k}',\underline{k}) = f_B(\underline{k}'-\underline{k}) + \int d^3\underline{q}_1 \int d^3\underline{q}_2 \, \tilde{V}(\underline{k}'-\underline{q}_1)\, G(\underline{q}_1,\underline{q}_2;k^2)\, \tilde{V}(\underline{q}_2-\underline{k}) \qquad (37)$$

where the Green's function G is given by

$$G(\underline{q}_1,\underline{q}_2;k^2) = -\frac{1}{4\pi}\int\frac{d^3\underline{r}_1}{(2\pi)^3}\int\frac{d^3\underline{r}_2}{(2\pi)^3}\, e^{-i\underline{q}_1\cdot\underline{r}_1} \langle \underline{r}_1 | \frac{1}{k^2+\nabla^2-V+i\epsilon} | \underline{r}_2 \rangle e^{i\underline{q}_2\cdot\underline{r}_2} \qquad (38)$$

Then using the representation (32) for the potential and some algebraic manipulation it follows that $g(E,t) \equiv f(E,t) - f_B(t)$ is regular inside an ellipse in the t -plane which intersects the real axis at $t = -4m^2$ and $t = 4m^2 + 4k^2$.

However, if we use the Fredholm solution of the Lippmann-Schwinger integral equation then it can be shown that each term in the Fredholm series is analytic in the t -plane cut from $-\infty$ to $-4m^2$ along the real axis and that the series is also uniformly convergent in any region of the t -plane excluding the cut. This means that the scattering amplitude, apart from the first Born approximation term is in fact analytic in the whole t -plane apart from the above-mentioned cut. In fact more than this: the analyticity may equally well be proved for g^*, and hence the analitic properties hold also for $\mathrm{Re}\, g$ and $\mathrm{Im}\, g$ separately.

The analyticity of f in the t -plane has also been studied by Klein[32], who uses essentially the methods of Klein and Zemach[28] and deduces a Mandelstam representation, Regge[33] whose method consists of introducing complex angular momenta into the Schrodinger equation and Bowcock and Martin[34] who, however, have only carried out the proof for each term in the Born series.

We now extend the dispersion relation obtained above, only for real $t \leq 4\alpha^2 \; (4m^2)$ to the whole t- plane apart from the cut. This dispersion relation was

$$\mathrm{Re}\, f(E,\tau) = \frac{1}{\pi} P \int_0^\infty \frac{\mathrm{Im}\, f(E',\tau)}{E'-E}\, dE' + \sum_{j=0}^{N} \frac{R_j(\tau)}{E-E_j} - \frac{\lambda}{4\pi} \tilde{V}(\tau) \qquad (30)$$

or, equivalently,

$$f(E,t) = f_B(t) + \sum_{j=0}^{N} \frac{R_j(t)}{E-E_j} + \frac{1}{\pi} \int_0^\infty \frac{\mathrm{Im}\, f(E',t)}{E'-E-i\epsilon}\, dE' \qquad (39)$$

But $\mathrm{Im}\, f(E',t)$ is, for real $E' \geq 0$, analytic in the cut t -plane; the residues $R_j(t)$ are polynomials of degree ℓ_j (the angular momentum of the j th bound state). It hence follows that the last two terms may be continued into the cut t -plane to define $g(E,t)$, analytic in the two complex variables E and t (apart from the cuts and the poles in the E plane) and this $g(E,t)$ can be shown to be identical with the actual $g(E,t)$.

So, to obtain an integral representation for $f(E,t)$ all we have to do is to obtain a dispersion relation for $\mathrm{Im}\, f(E',t)$ embodying the analyticity properties that have been mentioned above. The one difficulty is that we do not know the asymptotic properties of $\mathrm{Im}\, f(E',t)$ as $|t| \to \infty$ and the simplest assumption, that $\mathrm{Im}\, f(E',t) \to 0$ as $|t| \to \infty$ is inconsistent with the requirement of a unitary S-matrix (certainly if there are bound states, and perhaps even if not). We hence write, so as to cover all possible cases

$$\mathrm{Im}\, f(E',t) = \frac{t^{r+1}}{\pi} \int_0^\infty \frac{\rho(E',t')}{t'^{n+1}(t'+t)}\, dt' + \sum_{i=0}^{r} \frac{t^i}{i!}\, g_i(E') \tag{40}$$

where r is unspecified. The lower limit of the integral is written as 0 , but is actually never less than $4m^2$; the actual region in the E and t planes in which ρ does not vanish can be determined by unitarity requirements.

Hence we obtain a general form for the Mandelstam representation:

$$f(E,t) = f_B(t) + \sum_{j=1}^{n} \frac{R_j(t)}{E-E_j} + \sum_{i=0}^{r} t^r \int_0^\infty \frac{g_i(E')}{E'-E-i\epsilon}\, dE'$$
$$+ t^{r+1} \int_0^\infty \frac{dE'}{\pi} \int_0^\infty \frac{dt'}{\pi} \frac{\rho(E',t')}{t'^{n+1}(t'+t)(E'-E-i\epsilon)} \tag{41}$$

Blankenbecler et al. then show how to find the weight function ρ when there are no bound states by means of the unitarity condition. We use the assumption that $\mathrm{Im} f(E,t) \to 0$ as $|t| \to \infty$ which is quite consistent if there are no bound states and then the Mandelstam representation reduces to

$$f(E,t) = f_B(t) + \int_0^\infty \frac{dE'}{\pi} \int_0^\infty \frac{dt'}{\pi} \frac{\rho(E',t')}{(E'-E-i\epsilon)(t'+t)} \tag{42}$$

The unitarity condition is, in terms of the T -matrix $T^+T = -(T+T^+)$ and since $T_{\underline{k}\,\underline{k}'} = \frac{k}{2\pi i} f(\underline{k},\underline{k}')$ this condition reduces to

$$\mathrm{Im} f(E,t) = \frac{\sqrt{E}}{4\pi} \int d\Omega' \, f^*\left[E, (\underline{k}'-\underline{k}_1)^2\right] f\left[E, (\underline{k}_1-\underline{k})^2\right] \tag{43}$$

with $t = (\underline{k}'-\underline{k})^2$ and $\underline{k}'^2 = \underline{k}^2 = \underline{k}_1^2 = E$

This holds only for the physical region $t \leq 4E$, but meaning is given outside this region by analytic continuation. Then using equations (43) and (42) it is possible to construct an expression for ρ in a number of well-defined steps. If λ is the potential strength parameter, then it is possible to fill out the whole E,t plane, such that at any stage ρ is a polynomial in λ , the degree of which increases as we fill out the plane. In any finite region of the permitted E,t plane $\rho(E,t)$ is given exactly by a polynomial in λ . This leads to an expression for $f(E,t)$ which is the limit of a sequence of polynomials, and, if our assumption about $\mathrm{Im} f(E,t)$

as $|t| \to \infty$ is correct, this sequence will converge. If this assumption is not correct then we have to use a subtracted form of the Mandelstam representation

$$F(E,t) \equiv f(E,t) - f_0(t) - \sum_{i=0}^{r} \frac{t^i}{i!} g_i(E) = t^{r+1} \int \frac{dE'}{\pi} \int \frac{dt'}{\pi} \frac{\rho(E',t')}{(E'-E-i\epsilon)(t'+t)} \tag{44}$$

and find ρ in this case; the sequence of polynomials in λ will always converge to $F(E,t)$ provided there are enough subtractions. In order to know the scattering amplitude the functions $g_i(E)$ still need to be determined; these are not considered directly, but the partial waves are investigated separately.

Blankenbecler et al. also consider the case where we have one bound s-state and $\operatorname{Im} f(E,t) \to g(E) \neq 0$ as $|t| \to \infty$. Here, of course, we have the extra terms

$$\frac{R}{E-E_0} + \int_0^\infty \frac{dE'}{\pi} \frac{g(E')}{E'-E-i\epsilon} \tag{45}$$

in the Mandelstam representation. The term involving $g(E)$ is essential (if $R \neq 0$) in order to avoid contradiction of unitarity; it may be present even if there are no bound states. They show how to calculate the residue R , and an indication is given of a procedure whereby E_0 the binding energy may be determined.

As mentioned above, the partial waves are considered separately, and their analytic properties are investigated in order to obtain a dispersion relation. What we do is to write the usual expansion

$$f(E,t) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(E) P_\ell\left(1 - \frac{t}{2E}\right) \tag{46}$$

$$\text{with} \qquad f_\ell(E) = \frac{1}{\sqrt{E}} e^{i\delta_\ell} \sin\delta_\ell \tag{47}$$

We now project out partial waves from the Mandelstam representation for $f(E,t)$, using the familiar property of Legendre Polynomials:

$$\int_{-1}^{1} P_\ell(z) P_m(z)\, dz = \frac{2}{2\ell+1} \delta_{\ell m} \tag{48}$$

It is then possible just to read off the analytic properties of the partial wave amplitudes regarded as a function of E (there actually turns out to be a cut in both the positive and negative real axes) and using the asymptotic condition, which depends on the number of subtractions $(r+1)$ in the Mandelstam representation, we may obtain the required dispersion relations for the partial wave amplitudes. The f_ℓ for $\ell \geq r+1$ are hence determined by mere integration since the $\mathrm{Im}\, f_\ell$ are essentially determined by the weight function ρ, but for $\ell < r+1$, f_ℓ satisfies a non-linear integral equation, the handling of which has been discussed by Noyes and Wong.[35] What in fact we do is to write f_ℓ as the ratio of two functions, and a Fredholm type equation is obtained for both of these functions. This means that all the phase shifts are determined no matter how many subtractions we need to supply; or, in other words that the Mandelstam representation together with the requirement of unitarity is sufficient to specify completely the the non-relativistic scattering problem.

What this work has done is to succeed in deriving the Mandelstam representation using only the principles of non-relativistic quantum mechanics, in the case of a fairly wide class of potentials. It is not obvious that the method gives an insight into the field-theoretic case - indeed in the opinion of Blankenbecler et al. it does not (although Klein[32] hopes that his method of proving analyticity in the t- plane will have some application in field theory). There are several points where the comparison is closer, such as the iterative construction of the weight function (which we merely mentioned above) and the treatment of the analytic properties of the partial wave amplitudes. In any case, it is felt by Klein that the analogue of the dispersion relations for the partial wave amplitudes will have a more immediate application in field theory than the Mandelstam representation.

Questions to which the work has given no answer are

(i) What is the precise relation between the iterative procedure and the Born series?

(ii) how many subtractions are needed in any given case?

REFERENCES

(1) N.F. Mott and H.S.W. Massey: Theory of Atomic Collisions, (Oxford) 2nd Edition (1949).

(2) B.A. Lippmann and J. Schwinger. Phys. Rev. 79, 469 (1950).

(3) M. Gell-Mann and M.L. Goldberger. Phys. Rev. 91, 398 (1953).

(4) J.A. Wheeler. Phys. Rev. 52, 1107 (1937).

(5) W. Heisenberg. Z. Physik. 120, 513, 673 (1943).

(6) C. Møller. Dan. Mat. Fys. Medd. 23 No. 1 (1945).
ibid. 22 No. 19 (1946).

(7) J.M. Jauch. Helv. Phys. Acta 31, 127 (1958).

(8) J.M. Jauch. Helv. Phys. Acta 31, 661 (1958).

(9) H. Ekstein. Phys. Rev. 94, 1063 (1954).

(10) L.I. Schiff, 'Quantum Mechanics' (McGraw-Hill) 1st edition (1949).

(11) S. Sunakawa. Prog. Th. Phys. 14, 175 (1955).

(12) M.N. Hack. Phys. Rev. 96, 196 (1954).

(13) H. Ekstein. Phys. Rev. 101, 880 (1956).

(14) J. von Neumann, 'Mathematical Foundations of Quantum Mechanics' (Princeton University Press) (1955).

(15) F. Riesz and B. Sz. Nagy, 'Functional Analysis' (F. Ungar. Publishing Co., New York) (1955).

(16) M.H. Stone, 'Linear Transformations in Hilbert Space' Amer. Math. Soc. Coll. Publ. XV (1932).

(17) J.M. Cook. J. Math. Phys. (Massachusetts) 36, 82 (1957).

(18) J.M. Jauch and I.I. Zinnes. Nuovo Cimento 11, 553 (1959).

(19) N. Wiener, 'The Fourier Integral' (Cambridge, 1933) Section 15, Theorem 11.

(20) S.T. Kuroda, Nuovo Cimento 12, 431 (1959).

REFERENCES (CONTD).

(21) M.N. Hack. Nuovo Cimento 9, 731 (1958).

(22) T. Kato and S.T. Kuroda. Nuovo Cimento 14, 1102 (1959).

(23) I.I. Zinnes. Nuovo Cimento Suppl. 12, 87 (1959).

(24) A.G. Tixaire. Helv. Phys. Acta 32, 412 (1959).

(25) N.N. Khuri. Phys. Rev. 107, 1148 (1957).

(26) R. Jost and A. Pais. Phys. Rev. 82, 840 (1951).

(27) E.C. Titchmarsh. 'Theory of Functions' (Oxford)

(28) A. Klein and C. Zemach. Ann. Phys. 7, 440 (1959).

(29) S. Gasiorowicz and H.P. Noyes. Nuovo Cimento, 10, 78 (1958).

(30) N.N. Khuri and S.B. Treiman Phys. Rev. 109, 198 (1958).

(31) R. Blankenbecler, M.L. Goldberger, N.N. Khuri and S.B. Treiman. Ann. Phys. 10, 62 (1960).

(32) A. Klein. J. Math. Phys. (New York) 1, 41 (1960).

(33) T. Regge. Nuovo Cimento 14, 951 (1959).

(34) J. Bowcock and A. Martin. Nuovo Cimento 14, 516 (1959).

(35) H.P. Noyes and D.Y. Wong. Phys. Rev. Letters 3, 191 (1959).

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